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The Diffraction of Electric Waves Round a Perfectly Reflecting Obstacle

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IV. *The Diffraction of Electric Waves Round a Perfectly Reflecting Obstacle.*

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In a previous communication* it was verified that the effect at a point on a perfectly conducting sphere due to a Hertzian oscillator near to its surface was negligible in comparison with the effect that would have been produced at that point but for the presence of the sphere, when the point is at some distance from the oscillator and the radius of the sphere is large compared with the wave length of the oscillations. In what follows it is proposed to find the effect at all points produced by a Hertzian oscillator placed outside a conducting sphere whose radius is large compared with the wave length of the oscillations. For simplicity the axis of the oscillator will be assumed to pass through the centre of the sphere, but this assumption will not affect the generality of most of the results. An Appendix is added in which the more important mathematical relations required are established.

1. Let O be the centre of the conducting sphere of radius a, and let the oscillator be at a point O_1, the direction of the axis of the oscillator being OO_1, and the distance OO_1 being r_1. In this case the lines of magnetic force are circles which have the line OO_1 for common axis. If γ denotes the magnetic force at any point P, ρ the distance of P from OO_1, and z the distance of P from the plane through O perpendicular to OO_1, $\gamma\rho$ satisfies the differential equation

$$\frac{\partial^2}{\partial\rho^2}(\gamma\rho)-\frac{1}{\rho}\frac{\partial}{\partial\rho}(\gamma\rho)+\frac{\partial^2}{\partial z^2}(\gamma\rho)+\kappa^2\gamma\rho = 0,$$

where $2\pi/\kappa$ is the wave length of the oscillations. Transforming to polar co-ordinates (r, θ), where r is the distance OP and θ the angle POO_1, $z = r\cos\theta$ and $\rho = r\sin\theta$; hence, writing $\cos\theta = \mu$, the differential equation becomes

$$\frac{\partial^2}{\partial r^2}(\gamma\rho)+\frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial\mu^2}(\gamma\rho)+\kappa^2\gamma\rho = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

* 'Roy. Soc. Proc.,' vol. 72 (1904), p. 59.

The general solution of this equation is

$$\gamma\rho = r^{1/2} \sum_{1}^{\infty} \{A_n J_{n+1/2}(\kappa r) + B_n J_{-n-1/2}(\kappa r)\} (1-\mu^2) \frac{\partial P_n}{\partial \mu},$$

where $J_{n+1/2}(\kappa r)$ is BESSEL'S function of order $n+\frac{1}{2}$, and P_n is the zonal harmonic of order n.

If γ_1 is the magnetic force at the point P due to the oscillator, γ_1 is the real part of $C \frac{\partial}{\partial \rho} \frac{e^{-\iota\kappa(R-Vt)}}{R}$, where V is the velocity of radiation and R is the distance O_1P; this has to be expressed in the form given above for the solution of equation (1). Writing

$$\psi_1 = \rho \frac{\partial}{\partial \rho} \frac{e^{-\iota\kappa R}}{R},$$

where

$$R^2 = r^2 + r_1^2 - 2\mu r r_1,$$

it follows that

$$\psi_1 = \rho \frac{\partial}{\partial \rho} r^{-1/2} r_1^{-1/2} e^{1/4 \pi \iota} \sum_{0}^{\infty} (2n+1) e^{1/2 n \pi \iota} K_{n+1/2}(\iota\kappa r_1) J_{n+1/2}(\kappa r) P_n,$$

where $r_1 > r$, and

$$\psi_1 = \rho \frac{\partial}{\partial \rho} r^{-1/2} r_1^{-1/2} e^{1/4 \pi \iota} \sum_{0}^{\infty} (2n+1) e^{1/2 n \pi \iota} K_{n+1/2}(\iota\kappa r) J_{n+1/2}(\kappa r_1) P_n,$$

when $r > r_1$, where

$$K_{n+1/2}(\iota z) = \frac{\pi}{2 \cos n\pi} e^{-1/2(n+1/2)\pi\iota} \{J_{-n-1/2}(z) - e^{(n+1/2)\pi\iota} J_{n+1/2}(z)\}.$$

Now

$$\rho \frac{\partial}{\partial \rho} = (1-\mu^2) \left\{ r \frac{\partial}{\partial r} - \mu \frac{\partial}{\partial \mu} \right\},$$

$$(2n+1) \mu \frac{\partial P_n}{\partial \mu} = n \frac{\partial P_{n+1}}{\partial \mu} + (n+1) \frac{\partial P_{n-1}}{\partial \mu},$$

$$(2n+1) P_n = \frac{\partial P_{n+1}}{\partial \mu} - \frac{\partial P_{n-1}}{\partial \mu},$$

therefore, $r_1 > r$,

$$\psi_1 = e^{1/4\pi\iota} r_1^{-1/2} \sum_{1}^{\infty} \left[e^{1/2(n-1)\pi\iota} K_{n-1/2}(\iota\kappa r_1) \left\{ r \frac{\partial}{\partial r} r^{-1/2} J_{n-1/2}(\kappa r) - (n-1) r^{-1/2} J_{n-1/2}(\kappa r) \right\} \right.$$
$$\left. - e^{1/2(n+1)\pi\iota} K_{n+3/2}(\iota\kappa r_1) \left\{ r \frac{\partial}{\partial r} r^{-1/2} J_{n+3/2}(\kappa r) + (n+2) r^{-1/2} J_{n+3/2}(\kappa r) \right\} (1-\mu^2) \frac{\partial P_n}{\partial \mu} \right],$$

that is

$$\psi_1 = -\kappa r_1^{-1/2} r^{1/2} e^{1/4\pi\iota} \sum_{1}^{\infty} \{ e^{1/2(n-1)\pi\iota} K_{n-1/2}(\iota\kappa r_1) + e^{1/2(n+1)\pi\iota} K_{n+3/2}(\iota\kappa r_1) \} J_{n+1/2}(\kappa r) (1-\mu^2) \frac{\partial P_n}{\partial \mu},$$

or

$$\psi_1 = -r_1^{-3/2} r^{1/2} \sum_{1}^{\infty} (2n+1) e^{1/2(n+1/2)\pi\iota} K_{n+1/2}(\iota\kappa r_1) J_{n+1/2}(\kappa r) (1-\mu^2) \frac{\partial P_n}{\partial \mu} \quad . \quad . \quad (2).$$

Similarly, when $r > r_1$,

$$\psi_1 = -r_1^{-3/2} r^{1/2} \sum_1^\infty (2n+1)\, e^{1/2(n+1/2)\pi\iota} \mathrm{J}_{n+1/2}(\kappa r_1)\, \mathrm{K}_{n+1/2}(\iota\kappa r)\,(1-\mu^2)\frac{\partial \mathrm{P}_n}{\partial\mu} \quad . \quad . \quad (3).$$

The solution of equation (1) that is required is that solution ψ which becomes infinite in the same way as ψ_1 at the point $(r_1, 0)$ and for which $\partial\psi/\partial r$ vanishes when $r = a$; the real part of $\mathrm{C}\psi e^{\iota\kappa \mathrm{V}t}$ will be the required magnetic force at any point, for $\partial(\gamma\rho)/\partial r$ will vanish when $r = a$, that is, the electric force tangential to the sphere will vanish. From the expression (2) for ψ_1 it follows that the required solution is of the form

$$\psi = -r_1^{-3/2} r^{1/2} \sum_1^\infty (2n+1)\, e^{1/2(n+1/2)\pi\iota} \mathrm{K}_{n+1/2}(\iota\kappa r_1) \left\{\mathrm{J}_{n+1/2}(\kappa r) + \mathrm{B}_n \mathrm{K}_{n+1/2}(\iota\kappa r)\right\} (1-\mu^2)\frac{\partial \mathrm{P}_n}{\partial\mu},$$

where $r_1 > r > a$, and the constants are determined so that $\partial\psi/\partial r = 0$ when $r = a$, hence

$$\psi = -r_1^{-3/2} r^{1/2} \sum_1^\infty e^{1/2(n+1/2)\pi\iota} (2n+1)\, \mathrm{K}_{n+1/2}(\iota\kappa r_1)$$

$$\left\{ \mathrm{J}_{n+1/2}(\kappa r) - \frac{\frac{\partial}{\partial a}\{a^{1/2}\mathrm{J}_{n+1/2}(\kappa a)\}}{\frac{\partial}{\partial a}\{a^{1/2}\mathrm{K}_{n+1/2}(\iota\kappa a)\}} \mathrm{K}_{n+1/2}(\iota\kappa r) \right\} (1-\mu^2)\frac{\partial \mathrm{P}_n}{\partial\mu},$$

when

$$r_1 > r > a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

and

$$\psi = -r_1^{-3/2} r^{1/2} \sum_1^\infty (2n+1)\, e^{1/2(n+1/2)\pi\iota}$$

$$\left\{ \mathrm{J}_{n+1/2}(\kappa r_1) - \frac{\frac{\partial}{\partial a}\{a^{1/2}\mathrm{J}_{n+1/2}(\kappa a)\}}{\frac{\partial}{\partial a}\{a^{1/2}\mathrm{K}_{n+1/2}(\iota\kappa a)\}} \mathrm{K}_{n+1/2}(\iota\kappa r_1) \right\} \mathrm{K}_{n+1/2}(\iota\kappa r)\,(1-\mu^2)\frac{\partial \mathrm{P}_n}{\partial\mu},$$

when

$$r > r_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Writing

$$\kappa r = z, \qquad \kappa r_1 = z_1, \qquad \kappa a = z_0, \qquad z^{1/2}\mathrm{J}_{n+1/2}(z) = 2^{1/2}\pi^{-1/2}u,$$

$$z^{1/2}\mathrm{J}_{-n-1/2}(z) = (-)^n\, 2^{1/2}\pi^{-1/2}v, \qquad z_1^{1/2}\mathrm{J}_{n+1/2}(z_1) = 2^{1/2}\pi^{-1/2}u_1, \text{ \&c.},$$

$$u = \mathrm{R}^{1/2}\sin\phi, \qquad v = \mathrm{R}^{1/2}\cos\phi, \qquad u_1 = \mathrm{R}_1^{1/2}\sin\phi_1, \qquad v_1 = \mathrm{R}_1^{1/2}\cos\phi_1,$$

$$u_0 = \mathrm{R}_0^{1/2}\sin\phi_0, \qquad v_0 = \mathrm{R}_0^{1/2}\cos\phi_0,$$

it follows that

$$z^{1/2}\mathrm{K}_{n+1/2}(\iota z) = 2^{-1/2}\pi^{1/2}e^{-1/2(n+1/2)\pi\iota}(v - \iota u)$$

that is

$$z^{1/2}\mathrm{K}_{n+1/2}(\iota z) = 2^{-1/2}\pi^{1/2}\mathrm{R}^{1/2}e^{-1/2(n+1/2)\pi\iota - \phi\iota}$$

Again

$$\frac{\partial}{\partial z}\{z^{1/2}\mathrm{K}_{n+1/2}(\iota z)\} = 2^{-1/2}\pi^{1/2}e^{-1/2(n+1/2)\pi\iota-\phi\iota}\left\{\tfrac{1}{2}\mathrm{R}^{-1/2}\frac{\partial \mathrm{R}}{\partial z} - \iota \mathrm{R}^{1/2}\frac{\partial\phi}{\partial z}\right\};$$

now
$$v\frac{\partial u}{\partial z}-u\frac{\partial v}{\partial z}=1, \qquad \text{that is} \qquad R\frac{\partial\phi}{\partial z}=1,$$
therefore
$$\frac{\partial}{\partial z}\{z^{1/2}K_{n+1/2}(\iota z)\}=2^{-1/2}\pi^{1/2}\iota e^{-1/2(n+1/2)\pi\iota-\phi\iota}\left\{\tfrac{1}{2}\frac{\partial R}{\partial z}-\iota\right\}R^{-1/2},$$

and writing $\frac{1}{2}\frac{\partial R}{\partial z}=-\tan\chi$, this becomes
$$\frac{\partial}{\partial z}\{z^{1/2}K_{n+1/2}(\iota z)\}=-2^{-1/2}\pi^{1/2}e^{-1/2(n+1/2)\pi\iota-\phi\iota-\chi\iota}R^{-1/2}\sec\chi.$$

Similarly,
$$\frac{\partial}{\partial z}\{z^{1/2}J_{n+1/2}(z)\}=2^{1/2}\pi^{-1/2}R^{-1/2}\cos(\phi+\chi)\sec\chi,$$
and therefore
$$\frac{\partial}{\partial a}\{a^{1/2}J_{n+1/2}(\kappa a)\}\Big/\frac{\partial}{\partial a}\{a^{1/2}K_{n+1/2}(\iota\kappa a)\}=\frac{2}{\pi}e^{1/2(n+1/2)\pi\iota+(\phi_0+\chi_0)\iota}\cos(\phi_0+\chi_0).$$

Hence the relations (4) and (5) become
$$\psi=-\frac{1}{\kappa r_1^2}\sum_1^\infty(2n+1)R^{1/2}R_1^{1/2}e^{-\phi_1\iota}\{\sin\phi-\iota e^{(\phi_0+\chi_0-\phi)\iota}\cos(\phi_0+\chi_0)\}(1-\mu^2)\frac{\partial P_n}{\partial\mu},$$

when $r_1>r$, that is
$$\psi=\frac{\iota}{2\kappa r_1^2}\sum_1^\infty(2n+1)R_1^{1/2}R^{1/2}\{e^{(\phi-\phi_1)\iota}+e^{(2\phi_0+2\chi_0-\phi-\phi_1)\iota}\}(1-\mu^2)\frac{\partial P_n}{\partial\mu}. \quad . \quad . \quad (6),$$

when $r_1>r$, and
$$\psi=\frac{\iota}{2\kappa r_1^2}\sum_1^\infty(2n+1)R_1^{1/2}R^{1/2}\{e^{(\phi_1-\phi)\iota}+e^{(2\phi_0+2\chi_0-\phi-\phi_1)\iota}\}(1-\mu^2)\frac{\partial P_n}{\partial\mu}. \quad . \quad . \quad (7),$$
when $r>r_1$.

2. The value of ψ at any point will now be compared with the value of ψ_1 at the same point, and, the radius a of the sphere being supposed to be great compared with the wave-length $2\pi/\kappa$ of the oscillations, it will be sufficient to compare the principal parts of ψ and ψ_1. Now
$$\psi_1=\rho\frac{\partial}{\partial\rho}\frac{e^{-\iota\kappa R}}{R},$$
that is
$$\psi_1=(1-\mu^2)\left[r\frac{\partial}{\partial r}-\mu\frac{\partial}{\partial\mu}\right]\frac{e^{-\iota\kappa R}}{R};$$

hence the principal part $\bar{\psi}_1$ of ψ_1 is given by
$$\bar{\psi}_1=\frac{-\iota\kappa r^2}{R^2}(1-\mu^2)e^{-\iota\kappa R}.$$

In calculating the principal part of ψ it is convenient to consider first the contribution of the terms for which $n+\frac{1}{2}$ is greater than the least of the two quantities

z and z_1. In this case the approximations to be used for ϕ_0 and χ_0 are those given by relations (x.) of the Appendix ; from these

$$e^{2\iota(\phi_0+\chi_0)} = \frac{1+\iota\tan(\phi_0+\chi_0)}{1-\iota\tan(\phi_0+\chi_0)},$$

that is

$$e^{2\iota(\phi_0+\chi_0)} = -1+\iota e^{2T_0}+\&c.,$$

where T_0 is a large negative quantity. Hence

$$e^{\iota(\phi-\phi_1)}+e^{\iota(2\phi_0+2\chi_0-\phi-\phi_1)} = 2\iota\sin\phi e^{-\iota\phi_1}+\iota e^{2T_0}e^{-\iota(\phi+\phi_1)}+\&c.,$$

that is

$$e^{\iota(\phi-\phi_1)}+e^{\iota(2\phi_0+2\chi_0-\phi-\phi_1)} = \iota e^{2T}+\iota e^{2T_0}+\ldots$$

and

$$(2n+1)R^{1/2}R_1^{1/2}\{e^{\iota(\phi-\phi_1)}+e^{\iota(2\phi_0+2\chi_0-\phi-\phi_1)}\} = \frac{2n+1}{\{\sinh\delta\sinh\delta_1\}^{1/2}}\iota\{e^{T-T_1}+e^{2T_0-T-T_1}\}.$$

Now

$$r_1>r>a,$$

therefore

$$\delta_1<\delta<\delta_0 \quad\text{and}\quad -T_1<-T<-T_0;$$

hence the order of the corresponding term in ψ, compared with the principal part of ψ_1, is $(\kappa r)^{-1/2}$, and is multiplied by an exponential with negative exponent. Further, the portion of the series containing these terms is simply oscillatory on account of P_n, and therefore the sum of any number of these terms is not of higher order than $(\kappa r)^{-1/2}\bar{\psi}_1$ by the appendix ; hence the part contributed by the terms of the series, for which $n+\frac{1}{2}-z_1$ is of the same or higher order than $z_1^{1/3}$, is negligible compared with $\bar{\psi}_1$. Again, the terms for which $n+\frac{1}{2}-z$ is of the same or higher order than $z^{1/3}$, but $n+\frac{1}{2}-z_1$ is of lower order than $z_1^{1/3}$, are at most of the order $(\kappa r)^{-1/3}\bar{\psi}_1$, and therefore, as above, their sum is negligible in comparison with $\bar{\psi}_1$. Similar results hold when $r>r_1$. Hence the part of ψ, if any, which is of the same order as ψ_1, is contributed by terms for which $n+\frac{1}{2}$ is less than z or exceeds z by a quantity of lower order than $z^{1/3}$ when $r>r_1$, and by terms for which $n+\frac{1}{2}$ is less than z_1 or exceeds z_1 by a quantity of lower order than $z_1^{1/3}$ when $r>r_1$. Different treatment is necessary according to the form of approximate value of $P_n(\mu)$ that is appropriate.

3. When θ is small, the approximate value of $P_n(\mu)$ is given by

$$P_n(\mu) = J_0\{(2n+1)\sin\tfrac{1}{2}\theta\},$$

whence to the same order

$$\frac{\partial P_n}{\partial\mu} = (n+\tfrac{1}{2})\cos\tfrac{1}{2}\theta\operatorname{cosec}\theta J_1\{(2n+1)\sin\tfrac{1}{2}\theta\},$$

and the series for ψ is approximately

$$\psi = \frac{\iota}{\kappa r_1^2}\sin\theta\cos\tfrac{1}{2}\theta\sum_1(n+\tfrac{1}{2})^2R^{1/2}R_1^{1/2}\{e^{\iota(\phi-\phi_1)}+e^{\iota(2\phi_0+2\chi_0-\phi-\phi_1)}\}J_1\{(2n+1)\sin\tfrac{1}{2}\theta\},$$

when $r<r_1$.

Writing

$$S_1 = \frac{\iota}{\kappa r_1^2} \sin\theta \cos\tfrac{1}{2}\theta \sum_1 (n+\tfrac{1}{2})^2 R^{1/2} R_1^{1/2} e^{\iota(\phi-\phi_1)} J_1\{(2n+1)\sin\tfrac{1}{2}\theta\},$$

the appropriate approximations for R, R_1, ϕ, and ϕ_1 are

$$R = \sec\alpha, \quad \phi = z\cos\alpha - \tfrac{1}{2}n\pi + (n+\tfrac{1}{2})\alpha, \quad \text{where} \quad z\sin\alpha = n+\tfrac{1}{2},^*$$

$$R_1 = \sec\alpha_1, \quad \phi_1 = z_1\cos\alpha_1 - \tfrac{1}{2}n\pi + (n+\tfrac{1}{2})\alpha_1, \quad \text{where} \quad z_1\sin\alpha_1 = n+\tfrac{1}{2};^*$$

hence approximately

$$S_1 = \frac{\iota}{\kappa r_1^2} \sin\theta \cos\tfrac{1}{2}\theta \sum_1 (n+\tfrac{1}{2})^2 \sec^{1/2}\alpha \sec^{1/2}\alpha_1 e^{\iota[z\cos\alpha - z_1\cos\alpha_1 + (n+1/2)(\alpha-\alpha_1)]} . J_1\{(2n+1)\sin\tfrac{1}{2}\theta\}.$$

Remembering that the oscillations of $J_1\{(2n+1)\sin\frac{1}{2}\theta\}$ depend on $e^{\pm(2n+1)\iota\sin 1/2\theta}$, the principal part of S_1, arises from the terms in the neighbourhood of the term for which

$$\frac{d}{dn}\{z\cos\alpha - z_1\cos\alpha_1 + (n+\tfrac{1}{2})(\alpha-\alpha_1) \pm (2n+1)\sin\tfrac{1}{2}\theta\}$$

vanishes; for this $\alpha-\alpha_1$ is small since θ is small, and therefore, unless r is nearly equal to r_1, which means that the point is close to the oscillator, $n+\frac{1}{2}$ is small compared with z for the terms that contribute the principal part of S_1. Hence approximately

$$z\cos\alpha + (n+\tfrac{1}{2})\alpha = z + \frac{1}{2z}(n+\tfrac{1}{2})^2,$$

$$z_1\cos\alpha_1 + (n+\tfrac{1}{2})\alpha_1 = z_1 + \frac{1}{2z_1}(n+\tfrac{1}{2})^2,$$

and the principal part of S_1 is equal to the principal part of

$$\frac{\iota}{\kappa r_1^2} \sin\theta \cos\tfrac{1}{2}\theta \sum_1 (n+\tfrac{1}{2})^2 e^{\iota\left[z-z_1+1/2(n+1/2)^2\left(\frac{1}{z}-\frac{1}{z_1}\right)\right]} J_1\{(2n+1)\sin\tfrac{1}{2}\theta\};$$

that is, the principal part of S_1 is given by

$$\overline{S}_1 = \frac{\iota}{\kappa r_1^2} \sin\theta \cos\tfrac{1}{2}\theta e^{\iota(z-z_1)} \int_0^\infty e^{\frac{\iota}{2}\left(\frac{1}{z}-\frac{1}{z_1}\right)\zeta^2} \zeta^2 J_1(2\zeta\sin\tfrac{1}{2}\theta)\, d\zeta;$$

therefore

$$\overline{S}_1 = -\frac{\iota}{\kappa r_1^2} \sin\theta \cos\tfrac{1}{2}\theta e^{\iota(z-z_1)} \frac{2\sin\frac{1}{2}\theta}{(z^{-1}-z_1^{-1})^2} e^{-\iota\frac{2\sin^2 1/2\theta}{(z^{-1}-z_1^{-1})}},$$

or

$$\overline{S}_1 = -\frac{\iota\kappa r^2\sin^2\theta}{(r_1-r)^2} e^{-\iota\kappa\left(r_1-r+\frac{2rr_1}{r_1-r}\sin^2 1/2\theta\right)}.$$

* Appendix, Relations (ii).

Now the distance D from the oscillator to the point $(r_1\theta)$ is given by

$$D^2 = r^2+r_1^2-2rr_1\cos\theta,$$

whence retaining only the most important terms

$$\bar{S}_1 = -\frac{\iota\kappa r^2\sin^2\theta}{D^2}e^{-\iota\kappa D},$$

and therefore $\bar{S}_1 = \bar{\psi}_1$.

The same result follows when $r > r_1$. Again writing

$$S_2 = \frac{\iota}{\kappa r_1^2}\sin\theta\cos\tfrac{1}{2}\theta\sum_1(n+\tfrac{1}{2})^2 R^{1/2}R_1^{1/2}e^{\iota(2\phi_0+2\chi_0-\phi-\phi_1)}J_1\{(2n+1)\sin\tfrac{1}{2}\theta\},$$

the principal part of S_2 arises from the terms in the neighbourhood of the term for which

$$\frac{d}{dn}\{2\phi_0+2\chi_0-\phi-\phi_1\pm(2n+1)\sin\tfrac{1}{2}\theta\}$$

vanishes. When $n+\frac{1}{2}$ is less than z_0, this requires that $2\alpha_0-\alpha-\alpha_1$ is small, since θ is; when $n+\frac{1}{2}$ is greater than z_0, it requires that $\pi-\alpha-\alpha_1$ is small, that is α and α_1 are each nearly a right angle, which means that the point is close to the oscillator, thus the case of $2\alpha_0-\alpha-\alpha_1$ small need only be considered. In this case the principal part of S_2 is equal to the principal part of

$$\frac{\iota}{\kappa r_1^2}\sin\theta\cos\tfrac{1}{2}\theta\sum_1(n+\tfrac{1}{2})^2 e^{\iota[2z_0-z-z_1+1/2(n+1/2)^2(2z_0^{-1}-z^{-1}-z_1^{-1})]}J_1\{(2n+1)\sin\tfrac{1}{2}\theta\},$$

that is

$$\bar{S}_2 = \frac{\iota}{\kappa r_1^2}\sin\theta\cos\tfrac{1}{2}\theta\, e^{\iota(2z_0-z-z_1)}\int_0^\infty e^{1/2\iota(2z_0^{-1}-z^{-1}-z_1^{-1})\zeta^2}\zeta^2 J_1(2\zeta\sin\tfrac{1}{2}\theta)\,d\zeta,$$

whence

$$\bar{S}_2 = -\frac{\iota}{\kappa r_1^2}\sin\theta\cos\tfrac{1}{2}\theta\, e^{\iota(2z_0-z-z_1)}\frac{2\sin\frac{1}{2}\theta}{(2z_0^{-1}-z^{-1}-z_1^{-1})^2}e^{-\frac{2\iota\sin^2 1/2\theta}{(2z_0^{-1}-z^{-1}-z_1^{-1})}},$$

or

$$\bar{S}_2 = -\frac{\iota\kappa}{r^2}\sin^2\theta\,(2a^{-1}-r^{-1}-r_1^{-1})^{-2}e^{-\iota\kappa[r+r_1-2a+2\sin^2 1/2\theta(2a^{-1}-r^{-1}-r_1^{-1})^{-1}]}.$$

This represents waves reflected from the neighbourhood of the points Q on the sphere, which are such that O_1Q makes an angle α_0 with the radius OQ, and if D_1 is the distance O_1Q, D_2 the distance PQ, where P is the point (r,θ), the above becomes, after some reduction,

$$\bar{S}_2 = -\iota\kappa\sin^2\alpha_1 e^{-\iota\kappa(D_1+D_2)},$$

the result that would have been obtained by elementary methods. Hence the effect at a point P for which θ is small is the sum of the direct waves and the waves reflected at the surface of the sphere.

4. When θ is nearly equal to π the appropriate approximation for $P_n(\mu)$ is given by

$$P_n(\mu) = \cos n\pi J_0\{(2n+1)\sin\tfrac{1}{2}(\pi-\theta)\},$$

or, writing $\pi-\theta = \theta'$,

$$P_n(\mu) = \cos n\pi J_0\{(2n+1)\sin\tfrac{1}{2}\theta'\},$$

and

$$\frac{\partial P_n}{\partial\mu} = -(n+\tfrac{1}{2})\cos n\pi\cos\tfrac{1}{2}\theta'\operatorname{cosec}\theta' J_1\{(2n+1)\sin\tfrac{1}{2}\theta'\};$$

hence

$$\psi = -\frac{\iota}{\kappa r_1^2}\sin\theta'\cos\tfrac{1}{2}\theta'\sum_1(n+\tfrac{1}{2})^2 R^{1/2}R_1^{1/2}\cos n\pi\{e^{\iota(\phi-\phi_1)}+e^{\iota(2\phi_0+2\chi_0-\phi-\phi_1)}\}J_1\{(2n+1)\sin\tfrac{1}{2}\theta'\},$$

when $r<r_1$, where to the order required the terms for which $n+\frac{1}{2}$ exceeds z by a quantity of the same or of higher order than $z^{1/3}$ can be neglected. Writing

$$S_1 = -\frac{\iota}{\kappa r_1^2}\sin\theta'\cos\tfrac{1}{2}\theta'\sum_1(n+\tfrac{1}{2})^2\cos n\pi R^{1/2}R_1^{1/2}e^{\iota(\phi-\phi_1)}J_1\{(2n+1)\sin\tfrac{1}{2}\theta'\},$$

the terms of this series, for which $n+\frac{1}{2}$ is less than z, and $z-n-\frac{1}{2}$ is of higher order than $z^{1/3}$, are at most of the order $(\kappa r)^{3/2}(\kappa r_1)^{-2}\psi_1$, and therefore their sum will be negligible in comparison with ψ_1 unless $\frac{d}{dn}[\phi-\phi_1\pm n\pi\pm(2n+1)\sin\frac{1}{2}\theta']$ vanishes or is a multiple of 2π for some value of n in the series. Using the appropriate approximations for ϕ and ϕ_1, this requires that $\alpha-\alpha_1\pm\pi$ is small or nearly equal to a multiple of 2π since θ' is small, and this is impossible, for α and α_1 are each less than $\frac{1}{2}\pi$. The terms of the series S_1, for which $|z-n-\frac{1}{2}|$ is of lower order than $z^{1/3}$, are at most of the order $(\kappa r)^{5/3}(\kappa r_1)^{-2}\psi_1$, and their sum will be negligible in comparison with ψ_1 unless $\frac{d}{dn}[\phi-\phi_1\pm n\pi\pm(2n+1)\sin\frac{1}{2}\theta']$ is very small or nearly equal to a multiple of 2π for a value of n in this series; substituting the appropriate approximations for ϕ and ϕ_1, this requires that $\frac{1}{2}\pi-\alpha_1-\pi$ is small, which is impossible, for α_1 is less than $\frac{1}{2}\pi$. Hence S_1 is negligible in comparison with ψ_1. The same result follows when $r>r_1$. Again, writing

$$S_2 = -\frac{\iota}{\kappa r_1^2}\sin\theta'\cos\tfrac{1}{2}\theta'\sum_1(n+\tfrac{1}{2})^2\cos n\pi R^{1/2}R_1^{1/2}e^{\iota(2\phi_0+2\chi_0-\phi-\phi_1)}J_1\{(2n+1)\sin\tfrac{1}{2}\theta'\},$$

the terms of this series, for which $n+\frac{1}{2}$ is less than z_0, and $z_0-n-\frac{1}{2}$ is of the same or of higher order than $z_0^{1/3}$, are at most of the order $(\kappa a)^{3/2}(\kappa r_1)^{-2}\psi_1$, and therefore their sum will be negligible in comparison with ψ_1 unless

$$\frac{d}{dn}[2\phi_0+2\chi_0-\phi-\phi_1\pm n\pi\pm(2n+1)\sin\tfrac{1}{2}\theta']$$

vanishes, or is nearly equal to a multiple of 2π for some value of n in this series.

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Substituting for ϕ_0, χ_0, &c., their approximate values, this requires that $2\alpha_0-\alpha-\alpha_1\pm\pi$ is small, since θ' is small, and this is impossible for $\frac{1}{2}\pi>\alpha_0>\alpha$, $\frac{1}{2}\pi>\alpha_0>\alpha_1$; hence the sum of these terms is negligible in comparison with ψ_1. The terms of the series for which $|z_0-n-\frac{1}{2}|$ is of lower order than $z_0^{1/3}$ are of the order $(\kappa a)^{3/2}(\kappa r_1)^{-2}\psi_1$, and their sum is of highest order when $\frac{d}{dn}(2\phi_0+2\chi_0-\phi-\phi_1\pm n\pi)$ is small, or nearly equal to a multiple of 2π, for a value of n in this series. In this case the approximation for $\phi_0+\chi_0$ is $\frac{1}{3}\pi+\frac{1}{4c}3^{-1/2}(3\mu)^2$ [Appendix (vi.)], and the above condition becomes that $\pi-\alpha-\alpha_1\pm\pi$ is small, or nearly a multiple of 2π, which is satisfied if both α and α_1 are small. The sum of these terms then is

$$-\frac{\iota}{\kappa r_1^{\,2}}\sin\theta'\cos\tfrac{1}{2}\theta'\Sigma\,(n+\tfrac{1}{2})^2\,\mathrm{R}^{1/2}\mathrm{R}_1^{\,1/2}e^{\iota\,\frac{2\pi}{3}-z-z_1+1/2\,(n+1/2)^2\,(3^{1/6}2^{2/3}c^{-1}z^{-2/3}-z^{-1}-z_1^{-1})}\,\mathrm{J}_1\{(2n+1)\sin\tfrac{1}{2}\theta'\},$$

retaining only the most important part, and the sum is therefore of the order $(\kappa a)^{3/2+1/3}(\kappa r_1)^{-2}\psi_1$, which is of lower order than ψ_1, and therefore negligible in comparison with it. When $n+\frac{1}{2}$ is greater than z_0 and $n+\frac{1}{2}-z_0$ is of the same or of higher order than $z_0^{1/3}$, but $n+\frac{1}{2}$ is less than z and $z-n-\frac{1}{2}$ is of the same or of higher order than $z^{1/3}$, when $z<z_1$, the terms are of the order $(\kappa r)^{3/2}(\kappa r_1)^{-2}\psi_1$, and their sum will be of the same order as ψ_1, if $\frac{d}{dn}(2\phi_0+2\chi_0-\phi-\phi_1\pm n\pi)$ is small, or nearly a multiple of 2π, for a value of n in this series. This condition is satisfied if there is a value of n in the series for which $\pi-\alpha-\alpha_1\pm\pi$ is small, or nearly a multiple of 2π, which requires that both α and α_1 should be small. For the other values of n, which have to be taken account of, the order of the terms is $(\kappa r)^{5/3}(\kappa r_1)^{-2}\psi_1$, and the condition that their sum should be of the same order as ψ_1 is, that for some of these values of n, $\frac{1}{2}\pi-\alpha_1\pm\pi$ is small, which is impossible. Hence the part of the series S_2 which is of the order of ψ_1 arises from the terms beginning with a term for which n is equal to $z_0+\mathrm{A}z_0^{1/3}$, where A is a positive quantity of the order of unity, and the principal part of S_2 is therefore equal to the principal part of

$$-\frac{\iota}{2\kappa r_1^{\,2}}\sin\theta'\cos\tfrac{1}{2}\theta'\sum_{z_0+\mathrm{A}z_0^{1/3}}(n+\tfrac{1}{2})^2\mathrm{R}^{1/2}\mathrm{R}_1^{\,1/2}e^{\iota(2\phi_0+2\chi_0-\phi-\phi_1-n\pi)}\mathrm{J}_1\{(2n+1)\sin\tfrac{1}{2}\theta'\}$$

$$-\frac{\iota}{2\kappa r_1^{\,2}}\sin\theta'\cos\tfrac{1}{2}\theta'\sum_{z_0+\mathrm{A}z_0^{1/3}}(n+\tfrac{1}{2})^2\mathrm{R}^{1/2}\mathrm{R}_1^{\,1/2}e^{\iota(2\phi_0+2\chi_0-\phi-\phi_1+n\pi)}\mathrm{J}_1\{(2n+1)\sin\tfrac{1}{2}\theta'\}.$$

Now

$$e^{2\iota(\phi_0+\chi_0)}=\frac{1+\iota\tan(\phi_0+\chi_0)}{1-\iota\tan(\phi_0+\chi_0)},$$

that is [Appendix (x.)]

$$e^{2\iota(\phi_0+\chi_0)}=-1+\iota e^{2\mathrm{T}_0}\ldots,$$

where T_0 is a negative quantity and $-\mathrm{T}_0$ increases rapidly with n. Also, when n

and r_1 are so great compared with a that $\sin\alpha$ and $\sin\alpha_1$ can be replaced by α and α_1, the principal part of S_2 is equal to the principal part of

$$\frac{\iota}{\kappa r_1^2}\sin\theta'\cos\tfrac{1}{2}\theta' \sum_{z_0+Az_0^{1/3}} (n+\tfrac{1}{2})^2 e^{-\iota[z+z_1+1/2(n+1/2)^2(z^{-1}+z_1^{-1})]} J_1\{(2n+1)\sin\tfrac{1}{2}\theta'\},$$

which is equal to

$$\frac{\iota}{\kappa r_1^2}\sin\theta'\cos\tfrac{1}{2}\theta'\int_{z_0}^{\infty}\zeta^2 e^{-\iota[z+z_1+1/2\zeta^2(z^{-1}+z_1^{-1})]} J_1(2\zeta\sin\tfrac{1}{2}\theta')\,d\zeta.$$

When θ' is very small the value of the above integral is approximately

$$-\frac{\iota\kappa r^2}{(r+r_1)^2}\sin{}^2\theta' e^{-\iota\kappa(r+r_1)},$$

which is equal to the value of ψ_1 at the point. These results correspond exactly with those obtained by the usual treatment of the effect of an opaque circular screen interposed between a source of light and the point of observation, when the source and the point of observation are both at distances from the screen great compared with the radius of the screen. It has been assumed in the above approximation that $\dfrac{a^2}{\lambda r}+\dfrac{a^2}{\lambda r_1}$ is a small quantity whose square is negligible; when this condition is not satisfied the value of the integral involved depends on the quantity

$$\sqrt{(\kappa a)}\,(\alpha+\alpha_1)\,(\tan\alpha+\tan\alpha_1)^{-1/2},$$

where $\sin\alpha = a/r$, $\sin\alpha_1 = a/r_1$, and diminishes rapidly as this quantity increases.

5. When θ is not small or nearly equal to π, $P_n(\mu)$ can be replaced by its approximate value

$$2^{1/2}\pi^{-1/2}\{(n+\tfrac{1}{2})\sin\theta\}^{-1/2}\cos\{(n+\tfrac{1}{2})\theta-\tfrac{1}{4}\pi\},$$

whence

$$\frac{dP_n}{d\mu} = (2n+1)^{1/2}(\pi\sin^3\theta)^{-1/2}\sin\{(n+\tfrac{1}{2})\theta-\tfrac{1}{4}\pi\},$$

and therefore, when $r<r_1$,

$$\psi = \frac{\iota}{2\kappa r_1^2}\pi^{-1/2}(\sin\theta)^{1/2}\sum_1^\infty(2n+1)^{3/2}R^{1/2}R_1^{1/2}\{e^{\iota(\phi-\phi_1)}+e^{\iota(2\phi_0+2\chi_0-\phi-\phi_1)}\}\sin\{(n+\tfrac{1}{2})\theta-\tfrac{1}{4}\pi\},$$

approximately, that is

$$\psi = (2\pi)^{-1/2}(\sin\theta)^{1/2}(\kappa r_1^2)^{-1}\sum_1^\infty(n+\tfrac{1}{2})^{3/2}R^{1/2}R_1^{1/2}\{e^{\iota[\phi-\phi_1+(n+1/2)\theta-1/4\pi]}-e^{\iota[\phi-\phi_1-(n+1/2)\theta+1/4\pi]}$$
$$+e^{\iota[2\phi_0+2\chi_0-\phi-\phi_1+(n+1/2)\theta-1/4\pi]}-e^{\iota[2\phi_0+2\chi_0-\phi-\phi_1-(n+1/2)\theta+1/4\pi]}\},$$

which may be written

$$\psi = S_1-S_2+S_3-S_4,$$

where

$$S_1 = (2\pi)^{-1/2}(\sin\theta)^{1/2}(\kappa r_1^2)^{-1}\sum_1^\infty (n+\tfrac{1}{2})^{3/2} R^{1/2} R_1^{1/2} e^{\iota[\phi-\phi_1+(n+1/2)\theta-1/4\pi]},$$

$$S_2 = (2\pi)^{-1/2}(\sin\theta)^{1/2}(\kappa r_1^2)^{-1}\sum_1^\infty (n+\tfrac{1}{2})^{3/2} R^{1/2} R_1^{1/2} e^{\iota[\phi-\phi_1-(n+1/2)\theta+1/4\pi]},$$

$$S_3 = (2\pi)^{-1/2}(\sin\theta)^{1/2}(\kappa r_1^2)^{-1}\sum_1^\infty (n+\tfrac{1}{2})^{3/2} R^{1/2} R_1^{1/2} e^{\iota[2\phi_0+2\chi_0-\phi-\phi_1+(n+1/2)\theta-1/4\pi]},$$

$$S_4 = (2\pi)^{-1/2}(\sin\theta)^{1/2}(\kappa r_1^2)^{-1}\sum_1^\infty (n+\tfrac{1}{2})^{3/2} R^{1/2} R_1^{1/2} e^{\iota(2\phi_0+2\chi_0-\phi-\phi_1-(n+1/2)\theta+1/4\pi]}.$$

Since $r < r_1$, the formulæ (ii.) of the Appendix can be used for the terms of S_1 and S_2, for which $n+\frac{1}{2}$ is less than z and $z-(n+\frac{1}{2})$ is of the same or higher order than $z^{1/3}$, and the corresponding parts of S_1 and S_2 are given by

$$S_{11} = (2\pi)^{-1/2}(\sin\theta)^{1/2}(\kappa r_1^2)^{-1}\sum_1^{z-Az^{1/3}} (n+\tfrac{1}{2})^{3/2}\sec^{1/2}\alpha\sec^{1/2}\alpha_1 e^{\iota[z\cos\alpha-z_1\cos\alpha_1+(n+1/2)(\alpha-\alpha_1+\theta)-1/4\pi]},$$

$$S_{21} = (2\pi)^{-1/2}(\sin\theta)^{1/2}(\kappa r_1^2)^{-1}\sum_1^{z-Az^{1/3}} (n+\tfrac{1}{2})^{3/2}\sec^{1/2}\alpha\sec^{1/2}\alpha_1 e^{\iota[z\cos\alpha-z_1\cos\alpha_1+(n+1/2)(\alpha-\alpha_1-\theta)+1/4\pi]},$$

where $z\sin\alpha = z_1\sin\alpha_1 = n+\frac{1}{2}$ and A is of the order of unity.

Now

$$\frac{d}{dn}\{z\cos\alpha-z_1\cos\alpha_1+(n+\tfrac{1}{2})(\alpha-\alpha_1+\theta)\} = \alpha-\alpha_1+\theta,$$

and $\alpha-\alpha_1+\theta$ cannot be a small quantity for $\alpha > \alpha_1$, since $r_1 > r$; hence S_{11} is at most of the order $z_1^{-1/2}\psi_1$, and therefore negligible in comparison with ψ_1.

Again

$$\frac{d}{dn}\{z\cos\alpha-z_1\cos\alpha_1+(n+\tfrac{1}{2})(\alpha-\alpha_1-\theta)\} = \alpha-\alpha_1-\theta,$$

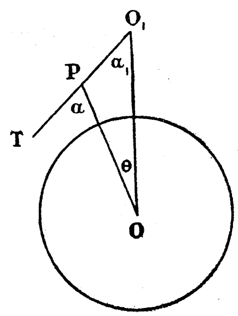

Fig. 1.

and $\alpha-\alpha_1-\theta$ will vanish for a value of n between 1 and $z-Az^{1/3}$, provided the point P (r, θ) lies inside the sphere described on OO_1 as diameter, but not close to its boundary. If α and α_1 now correspond to the value of n for which $\theta = \alpha-\alpha_1$, α is the angle OPT and α_1 the angle OO_1P in the figure,* and the principal part of S_{21} is given by

$$\bar{S}_{21} = (2\pi)^{-1/2}\sin^{1/2}\theta(\kappa r_1^2)^{-1}(z_1\sin\alpha_1)^{3/2}\sec^{1/2}\alpha\sec^{1/2}\alpha_1 e^{\iota(z\cos\alpha-z_1\cos\alpha_1+1/4\pi)}\int_{-\mu}^{\mu} e^{1/2\iota\zeta^2\left(\frac{1}{z\cos\alpha}-\frac{1}{z_1\cos\alpha_1}\right)}d\zeta,$$

where μ is large, that is

$$\bar{S}_{21} = (2\pi)^{-1/2}\sin^{1/2}\theta(\kappa r_1^2)^{-1}(z_1\sin\alpha_1)^{3/2}\sec^{1/2}\alpha\sec^{1/2}\alpha_1 e^{\iota(z\cos\alpha-z_1\cos\alpha_1+1/4\pi)}(2\pi)^{1/2}e^{1/4\pi\iota}\left\{\frac{1}{z\cos\alpha}-\frac{1}{z_1\cos\alpha_1}\right\}^{-1/2}.$$

* The corresponding value of n is given by $n+\frac{1}{2} = \kappa p$, where p is the perpendicular from O on O_1P.

If $O_1P = D$, then

$$r_1 \cos \alpha_1 - r \cos \alpha = D, \qquad r \sin \theta = D \sin \alpha_1,$$

and therefore

$$\bar{S}_{21} = \iota\kappa \sin^2 \theta r^2 D^{-2} e^{-\iota\kappa D},$$

that is for the points defined above $\bar{S}_{21} = -\bar{\psi}_1$.

The same formulæ (ii.) of the Appendix are applicable to the terms of the series S_3 and S_4, for which $n+\frac{1}{2}$ is less than z_0 and $z_0-(n+\frac{1}{2})$ is of the same or higher order than $z_0^{1/3}$. The corresponding parts of the series S_3 and S_4 are given by

$$S_{31} = (2\pi)^{-1/2} \sin^{1/2}\theta (\kappa r_1^2)^{-1} \sum_{1}^{z_0 - Az_0^{1/3}} (n+\tfrac{1}{2})^{3/2} \sec^{1/2}\alpha \sec^{1/2}\alpha_1 e^{\iota[2z_0 \cos\alpha_0 - z\cos\alpha - z_1 \cos\alpha + (n+1/2)(2\alpha_0 - \alpha - \alpha_1 + \theta) - 1/4\pi]},$$

$$S_{41} = (2\pi)^{-1/2} \sin^{1/2}\theta (\kappa r_1^2)^{-1} \sum_{1}^{z_0 - Az_0^{1/3}} (n+\tfrac{1}{2})^{3/2} \sec^{1/2}\alpha \sec^{1/2}\alpha_1 e^{\iota[2z_0 \cos\alpha_0 - z\cos\alpha - z_1 \cos\alpha_1 + (n+1/2)(2\alpha_0 - \alpha - \alpha_1 - \theta) + 1/4\pi]},$$

where $z_0 \sin \alpha_0 = z \sin \alpha = z_1 \sin \alpha_1 = n+\frac{1}{2}$. Now

$$\frac{d}{dn}\{2z_0 \cos\alpha_0 - z\cos\alpha - z_1\cos\alpha + (n+\tfrac{1}{2})(2\alpha_0 - \alpha - \alpha_1 + \theta)\} = 2\alpha_0 - \alpha - \alpha_1 + \theta,$$

and $\alpha_0 > \alpha$ for $r > a$, $\alpha_0 > \alpha_1$ for $r_1 > a$, hence $2\alpha_0 - \alpha - \alpha_1 + \theta$ is always finite, and therefore, as above, the sum of the terms S_{31} is of lower order than ψ_1. Also

$$\frac{d}{dn}\{2z_0 \cos\alpha_0 - z\cos\alpha - z_1\cos\alpha_1 + (n+\tfrac{1}{2})(2\alpha_0 - \alpha - \alpha_1 - \theta)\} = 2\alpha_0 - \alpha - \alpha_1 - \theta,$$

and $2\alpha_0 - \alpha - \alpha_1 - \theta$ will vanish for a value of n between 1 and $z_0 - Az_0^{1/3}$, provided the point $P(r, \theta)$ satisfies conditions to be determined. For let $n_0 + \frac{1}{2} = \kappa p$, where $p < a$, and let O_1Q be a straight line through O_1 at a distance p from O, meeting the sphere in Q (fig. 2), then the angle TQO_1 is α_0, where $n_0+\frac{1}{2} = z_0 \sin \alpha_0$, and the angle OO_1Q is α_1, where $n_0 + \frac{1}{2} = z_1 \sin \alpha_1$. Hence, drawing QP in the plane O_1OQ making the angle TQP equal to α_0, and taking any point P on QP, the angle QPO is equal to α, where $n_0+\frac{1}{2} = z \sin \alpha$, and the angle POO_1, which is θ, is equal to $2\alpha_0 - \alpha - \alpha_1$. For any position of P on the line QP the terms of the series S_{41} in the neighbourhood of the terms for which $n = n_0$ contribute a sum which is of higher order than any of them, and the principal part of S_{41} is given by

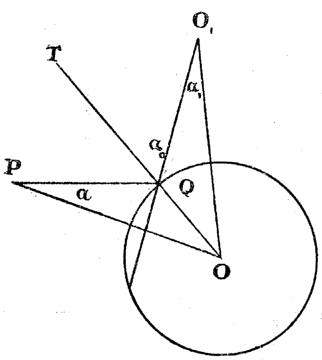

Fig. 2.

$$\bar{S}_{41} = (2\pi)^{-1/2} \sin^{1/2}\theta (n_0+\tfrac{1}{2})^{3/2} \sec^{1/2}\alpha \sec^{1/2}\alpha_1 e^{\iota(2z_0\cos\alpha_0 - z\cos\alpha - z_1\cos\alpha_1 + 1/4\pi)}$$

$$\int_{-\mu}^{\mu} e^{1/2\iota\zeta^2\left(\frac{2}{z_0\cos\alpha_0} - \frac{1}{z\cos\alpha} - \frac{1}{z_1\cos\alpha_1}\right)} d\zeta,$$

whence

$$\bar{S}_{41} = \iota(\kappa r_1^2)^{-1} \sin^{1/2}\theta \, (n_0+\tfrac{1}{2})^{3/2} \sec^{1/2}\alpha \sec^{1/2}\alpha_1 \left(\frac{2}{z_0\cos\alpha_0} - \frac{1}{z\cos\alpha} - \frac{1}{z_1\cos\alpha_1}\right)^{-1/2} e^{\iota(2z_0\cos\alpha_0 - z\cos\alpha - z_1\cos\alpha_1)},$$

that is

$$\bar{S}_{41} = \iota\kappa \sin^2\alpha_1 \sin^{1/2}\theta \, (z_1\sin\alpha_1)^{-1/2} \sec^{1/2}\alpha \sec^{1/2}\alpha_1 \left(\frac{2}{z_0\cos\alpha_0} - \frac{1}{z\cos\alpha} - \frac{1}{z_1\cos\alpha_1}\right)^{-1/2} e^{\iota(2z_0\cos\alpha_0 - z\cos\alpha - z_1\cos\alpha_1)},$$

which after some reduction can be put in the form

$$\bar{S}_{41} = \iota\kappa \sin^2\alpha_1 \left(\frac{2\cos\alpha_0}{a} + \frac{1}{QP} + \frac{1}{O_1Q}\right)^{1/2} \left(\frac{2}{a\cos\alpha_0} + \frac{1}{QP} + \frac{1}{O_1Q}\right)^{-1/2} e^{-\iota\kappa(O_1Q+QP)},$$

and this represents the effect at P of the waves reflected from the sphere at Q.*

From the diagram it appears that the points P for which the condition $\theta = 2\alpha_0 - \alpha - \alpha_1$ can be satisfied are the points outside the tangent cone drawn from O_1 to the sphere (that is the points outside the geometrical shadow), and as P approaches the surface of the cone the value of $n_0+\frac{1}{2}$ approaches κa; hence the above result holds for all points outside the tangent cone and not close to its boundary.

Omitting for the present the discussion of the terms of the series S_3 and S_4 for

* This result can be verified by elementary methods as follows: If M is the amplitude of the magnetic force at Q, M_1 the amplitude of the magnetic force at P in the waves reflected from Q, the intensity at Q in a beam of rays from O_1 of small cross section is

$$M^2a^2 \sin(\alpha_0 - \alpha_1)\cos\alpha_0 \, d(\alpha_0 - \alpha_1)\, d\phi,$$

and the intensity in the same beam at P is

$$M_1^2 r^2 \sin\theta \cos\alpha \, d\theta \, d\phi,$$

and these intensities are equal, therefore

$$M^2a^2 \sin(\alpha_0 - \alpha_1)\cos\alpha_0 \, d(\alpha_0 - \alpha_1) = M_1^2 r^2 \sin\theta\cos\alpha\, d\theta;$$

now

$$\theta = 2\alpha_0 - \alpha - \alpha_1,$$

$$z_0\cos\alpha_0\, d\alpha_0 = z\cos\alpha\, d\alpha = z_1\cos\alpha_1\, d\alpha_1,$$

$$r_1\cos\alpha_1 - a\cos\alpha_0 = O_1Q = a\sin(\alpha_0-\alpha_1)/\sin\alpha_1,$$

whence

$$M_1^2 r^2 \sin{}^2\theta = Ma^2\sin^2(\alpha_0-\alpha_1)\sin\theta\sec\alpha\sec\alpha_1 \Big/ z_1\sin\alpha_1\left(\frac{2}{z_0\cos\alpha_0} - \frac{1}{z\cos\alpha} - \frac{1}{z_1\cos\alpha_1}\right).$$

The amplitude of the ψ of the waves from O_1 at Q is $Ma\sin(\alpha_0-\alpha_1)$, and of the reflected waves at P is $M_1 r\sin\theta$, therefore the ψ of the reflected waves at P is

$$-\iota\kappa\sin^2\alpha_1\sin^{1/2}\theta\sec^{1/2}\alpha\sec^{1/2}\alpha_1(z_1\sin\alpha_1)^{-1/2}\left(\frac{2}{z_0\cos\alpha_0} - \frac{1}{z\cos\alpha} - \frac{1}{z_1\cos\alpha_1}\right)^{-1/2},$$

which is the result obtained from the series.

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which $n+\frac{1}{2}-z_0$ is of lower order than $z_0^{1/3}$,* the terms for which $z-n-\frac{1}{2}$ is of the same or of higher order than $z^{1/3}$ will now be considered. Denoting these terms of the series S_3 and S_4 by S_{32} and S_{42}, and remembering that

$$e^{2\iota(\phi_0+\chi_0)} = -1+\iota e^{2T_0} \ldots, \text{ \&c.},$$

where T_0 is a negative real quantity increasing rapidly with n, S_{32} and S_{42} are to the required order of approximation given by

$$S_{32} = -(2\pi)^{-1/2}(\kappa r_1^2)^{-1}\sin^{1/2}\theta \sum^{z-Az^{1/3}} (n+\tfrac{1}{2})^{3/2}\sec^{1/2}\alpha\sec^{1/2}\alpha_1 e^{\iota[n\pi-z\cos\alpha-z_1\cos\alpha_1-(n+1/2)(\alpha+\alpha_1-\theta)-1/4\pi]},$$

$$S_{42} = -(2\pi)^{-1/2}(\kappa r_1^2)^{-1}\sin^{1/2}\theta \sum^{z-Az^{1/3}} (n+\tfrac{1}{2})^{3/2}\sec^{1/2}\alpha\sec^{1/2}\alpha_1 e^{\iota[n\pi-z\cos\alpha-z_1\cos\alpha_1-(n+1/2)(\alpha+\alpha_1+\theta)+1/4\pi]}.$$

The value of $\frac{d}{dn}\{n\pi-z\cos\alpha-z_1\cos\alpha_1-(n+\frac{1}{2})(\alpha+\alpha_1-\theta)\}$ is $\pi-\alpha-\alpha_1+\theta$, which, since α and α_1 are less than $\frac{1}{2}\pi$ cannot be very small, and therefore the sum of the terms S_{32} is of lower order than ψ_1. Again the value of

$$\frac{d}{dn}\{n\pi-z\cos\alpha-z_1\cos\alpha_1-(n+\tfrac{1}{2})(\alpha+\alpha_1+\theta)\}$$

is $\pi-\alpha-\alpha_1-\theta$, and, if P is a point outside the geometrical shadow for which the angle OPO_1 is less than a right angle, there is a value n_1 of n in the series S_{42} for which $\pi-\alpha-\alpha_1-\theta$ vanishes, viz., that given by $n_1+\frac{1}{2}=\kappa p$ where p is the perpendicular from O on O_1P, and the principal part of S_{42} is given by

$$\overline{S}_{42} = -(2\pi)^{-1/2}(\kappa r_1^2)^{-1}(n_1+\tfrac{1}{2})^{3/2}\sec^{1/2}\alpha\sec^{1/2}\alpha_1 e^{-\iota(z\cos\alpha+z_1\cos\alpha_1+1/4\pi)}\int_{-\mu}^{\mu} e^{-1/2\iota\zeta^2\left(\frac{1}{z\cos\alpha}+\frac{1}{z_1\cos\alpha_1}\right)}d\zeta,$$

where

$$n_1+\tfrac{1}{2} = z\sin\alpha = z_1\sin\alpha_1,$$

that is by

$$\overline{S}_{42} = -(2\pi)^{-1/2}(\kappa r_1^2)^{-1}(n_1+\tfrac{1}{2})^{3/2}\sec^{1/2}\alpha\sec^{1/2}\alpha_1 e^{-\iota(z\cos\alpha+z_1\cos\alpha_1+1/2\pi)}(2\pi)^{1/2}\left(\frac{1}{z\cos\alpha}+\frac{1}{z_1\cos\alpha_1}\right)^{-1/2},$$

or

$$\overline{S}_{42} = \iota\kappa\sin^2\alpha_1 e^{-\iota\kappa D},$$

where D is the distance O_1P. Hence

$$\overline{S}_{42} = -\overline{\psi}_1.$$

The terms of the series S_1, S_2, S_3, S_4 for which $|n+\frac{1}{2}-z|$ is of lower order than $z^{1/3}$ and $z_1-n-\frac{1}{2}$ is of the same or higher order than $z_1^{1/3}$ are of the order $z^{-1/3}\psi_1$, and, that their

* It will be proved below that these terms are important only in the neighbourhood of the geometrical shadow.

sum should be of the order of ψ_1, $\frac{1}{2}\pi-\alpha_1-\theta$ must be small, that is, the point P must be on or close to the boundary of the sphere described on OO_1 as diameter. In these terms of the series S_3 and S_4, $e^{2\iota(\phi_0+\chi_0)}$ can be replaced by -1, and then the sum of the terms of the series $S_1-S_2+S_3-S_4$ required is the same as the sum of the corresponding terms of the series

$$\frac{\iota}{2\kappa r_1^{\,2}}\sum_1^\infty (2n+1)\,R^{1/2}R_1^{\,1/2}\{e^{\iota(\phi-\phi_1)}-e^{-\iota(\phi+\phi_1)}\}\,(1-\mu^2)\frac{dP_n}{d\mu}.$$

Now, for the values of θ specified above, $\frac{d}{dn}\{\pm\phi-\phi_1\pm(n+\frac{1}{2})\theta\}$ can only be small when α is a right angle or differs from a right angle by a small quantity, hence the principal part of the sum of the terms of the series $S_1-S_2+S_3-S_4$ is the same as the principal part of ψ_1, that is, it is equal to $\bar{\psi}_1$.

6. When $r > r_1$, ϕ and ϕ_1 are interchanged, the value of ψ may be written

$$\psi = S'_1-S'_2+S_3-S_4,$$

and the discussion of the series is identical with that of the previous case. If a plane be drawn through O_1 perpendicular to OO_1, then for points P on the side of this plane remote from the sphere and not close to the plane the principal part of S'_2 is $\bar{\psi}_1$; for points outside the geometrical shadow and not close to its boundary the terms of the series S_4, for which n is less than z_0, have a sum of the same order as ψ_1, which represents the effect of reflected waves; for points outside the geometrical shadow between the plane through O_1 and the sphere centre O, radius OO_1, the terms of the series S_4, for which n is greater than z_0, have a sum whose principal part is $-\bar{\psi}_1$, and for points in the neighbourhood of the plane through O_1 the terms of ψ, for which $|n+\frac{1}{2}-z_1|$ is of lower order than $z_1^{\,1/3}$, have a sum whose principal part is $\bar{\psi}_1$. Thus for all points outside the geometrical shadow and not close to its boundary the principal part of the value of ψ obtained from the series is the same as that which would be given by applying the methods of geometrical optics.

7. The effect of the terms of the series S_3 and S_4, beginning with those for which $z_0-n-\frac{1}{2}$ is of the same or of lower order than $z_0^{\,1/3}$, has now to be obtained. The corresponding part of S_4 may be written

$$S_{43} = (2\pi)^{-1/2}(\sin\theta)^{1/2}(\kappa r_1^{\,2})^{-1}\sum_{z_0-Az_0^{1/3}}(n+\tfrac{1}{2})^{3/2}R^{1/2}R_1^{\,1/2}e^{\iota[2\phi_0+2\chi_0-\phi-\phi_1-(n+1/2)\theta+1/4\pi]},$$

now for values of n greater than z_0 it follows, from the corresponding expression for $2\phi_0+2\chi_0$ [Appendix (vi.)], that the factor $e^{2\iota(\phi_0+\chi_0)}$ does not oscillate but approaches the value -1 as n increases, hence for these values of n it may be written

$$e^{2\iota(\phi_0+\chi_0)} = -1-2\sum_1^\infty(-\iota)^k\cot^k(\phi_0+\chi_0).$$

Writing

$$u_0 = w_0 e^{T_0}, \qquad v_0 = w_0 e^{-T_0},$$

and remembering that

$$\cot(\phi_0+\chi_0) = -u'_0/v'_0,$$

it follows that

$$\cot(\phi_0+\chi_0) = \frac{1+2w_0w'_0}{1-2w_0w'_0}\, e^{2T_0},$$

where $(1+2w_0w'_0)/(1-2w_0w'_0)$ differs from unity by a small quantity for all values of n greater than z_0, and T_0 is a negative quantity increasing numerically with n; whence

$$e^{2\iota(\phi_0+\chi_0)} = -1-2\sum_1^\infty (-\iota)^k \left(\frac{1+2w_0w'_0}{1-2w_0w'_0}\right)^k e^{2kT_0}$$

and

$$S_{43} = (2\pi)^{-1/2}(\sin\theta)^{1/2}(\kappa r_1^{\,2})^{-1}\Bigg[\sum_{z_0-Az_0^{1/3}}^{z_0}(n+\tfrac{1}{2})^{3/2}R^{1/2}R_1^{\,1/2}e^{\iota[2\phi_0+2\chi_0-\phi-\phi_1-(n+1/2)\theta+1/4\pi]}$$
$$-\sum_{z_0}(n+\tfrac{1}{2})^{3/2}R^{1/2}R_1^{\,1/2}e^{-\iota[\phi+\phi_1+(n+1/2)\theta-1/4\pi]}$$
$$-2\sum_{z_0}\sum_1^\infty(n+\tfrac{1}{2})^{3/2}R^{1/2}R_1^{\,1/2}\left(\frac{1+2w_0w'_0}{1-2w_0w'_0}\right)^k(-\iota)^k e^{2kT_0-\iota[\phi+\phi_1+(n+1/2)\theta-1/4\pi]}\Bigg].$$

In the series

$$\sum_{z_0}\sum_1^\infty(n+\tfrac{1}{2})^{3/2}R^{1/2}R_1^{\,1/2}\left(\frac{1+2w_0w'_0}{1-2w_0w'_0}\right)^k(-\iota)^k e^{2kT_0-\iota[\phi+\phi_1+(n+1/2)\theta-1/4\pi]},$$

writing $n+\frac{1}{2} = z_0+\zeta$, since $2T_0 = \log\tan\phi_0$, the coefficient of ζ in $2T_0$ is of the order $z_0^{-1/3}$; hence, if the value of

$$\theta+\frac{\partial\phi}{\partial n}+\frac{\partial\phi_1}{\partial n}$$

where $n+\frac{1}{2} = z_0$ is of higher order than $z_0^{-1/3}$, the sum of this double series is of lower order than its first term multiplied by $z_0^{1/3}$, and therefore to the order required negligible. Again, in the series

$$\sum_{z_0-Az_0^{1/3}}^{z_0}(n+\tfrac{1}{2})^{3/2}R^{1/2}R_1^{\,1/2}e^{\iota[2\phi_0+2\chi_0-\phi-\phi_1-(n+1/2)\theta+1/4\pi]}$$

the factor $e^{2\iota(\phi_0+\chi_0)}$ oscillates and, writing $n+\frac{1}{2} = z_0-\zeta$,

$$2\phi_0+2\chi_0 = \tfrac{1}{3}\pi+3^{-1/2}(4c)^{-1}(3\mu)^2$$

[see Appendix (vi.)], where

$$3\mu = -6^{1/3}z_0^{-1/3}\zeta, \qquad c = 2^{-1/3}\pi^{1/2}\Pi(-\tfrac{1}{6}),$$

and therefore the sum of this series is of lower order than its last term multiplied by $z_0^{1/3}$ when the value of

$$\theta+\frac{\partial\phi}{\partial n}+\frac{\partial\phi_1}{\partial n}$$

for $n+\frac{1}{2}=z_0$ is of higher order than $z_0^{1/3}$. Hence the principal part of S_{43} is equal to the principal part of

$$-(2\pi)^{-1/2}(\sin\theta)^{1/2}(\kappa r_1^2)^{-1}\sum_{z_0}(n+\tfrac{1}{2})^{3/2}R^{1/2}R_1^{1/2}e^{-\iota[\phi+\phi_1+(n+1/2)\theta-1/4\pi]}$$

when the value of

$$\theta+\frac{\partial\phi}{\partial n}+\frac{\partial\phi_1}{\partial n}$$

for $n+\frac{1}{2}=z_0$ is of higher order than $z_0^{-1/3}$.*

Now

$$-(2\pi)^{-1/2}(\sin\theta)^{1/2}(\kappa r_1^2)^{-1}\sum_{z_0}(n+\tfrac{1}{2})^{3/2}R^{1/2}R_1^{1/2}e^{-\iota[\phi+\phi_1+(n+1/2)\theta-1/4\pi]}$$
$$=(2\pi)^{-1/2}(\sin\theta)^{1/2}(\kappa r_1^2)^{-1}\left[\sum_1^{z_0}(n+\tfrac{1}{2})^{3/2}R^{1/2}R_1^{1/2}e^{-\iota[\phi+\phi_1+(n+1/2)\theta-1/4\pi]}\right.$$
$$\left.-\sum_1(n+\tfrac{1}{2})^{3/2}R^{1/2}R_1^{1/2}e^{-\iota[\phi+\phi_1+(n+1/2)\theta-1/4\pi]}\right];$$

the principal part of the second series has been found above, and arises from the terms on both sides of the term for which

$$\theta+\frac{\partial\phi}{\partial n}+\frac{\partial\phi_1}{\partial n}=0.$$

If n_1 is the corresponding value of n, and

$$n_1+\tfrac{1}{2}=z\sin\alpha=z_1\sin\alpha_1,$$

n_1 is given by

$$\theta=\pi-\alpha-\alpha_1,$$

hence, writing $n=n_1+\zeta$, the principal part of the above series is

$$(2\pi)^{-1/2}(\sin\theta)^{1/2}(\kappa r_1^2)^{-1}(n_1+\tfrac{1}{2})^{3/2}\sec\alpha\sec^{1/2}\alpha_1 e^{-\iota(z\cos\alpha+z_1\cos\alpha_1+1/4\pi)}$$
$$\left[\int_{-\infty}^{z_0-n_1-1/2}e^{-1/2\iota\zeta^2\left(\frac{1}{z\cos\alpha}+\frac{1}{z_1\cos\alpha_1}\right)}d\zeta-\int_{-\infty}^{\infty}e^{-1/2\iota\zeta^2\left(\frac{1}{z\cos\alpha}+\frac{1}{z_1\cos\alpha_1}\right)}d\zeta\right].$$

Therefore, if D denotes the distance from O_1 to the point P (r,θ),

$$\sin\theta/D=\sin\alpha/r_1=\sin\alpha/r,$$

and making the substitution

$$\zeta=\pi^{1/2}\left(\frac{1}{z\cos\alpha}+\frac{1}{z_1\cos\alpha_1}\right)^{-1/2}\eta,$$

* If ϵ is an angle of higher order than $z_0^{-1/3}$, γ the semivertical angle of the tangent cone to the sphere from O_1, and the tangent cones with vertices on OO_1 whose semivertical angles are $\gamma\pm\epsilon$ are constructed, the space excluded is that between these two tangent cones, which are both close to the boundary of the geometrical shadow.

the above expression becomes

$$-2^{-1/2}\kappa \sin^2 \alpha_1 e^{-\iota\kappa D - 1/4\pi\iota} \int_{\eta_0}^{\infty} e^{-1/2\pi\eta^2\iota}\, d\eta,$$

where

$$\eta_0 = (a - r \sin \alpha)\,(2D)^{1/2}\,(\lambda r r_1 \cos \alpha \cos \alpha_1)^{-1/2},$$

hence

$$\overline{S}_{43} = -2^{-1/2}\overline{\psi}_1 e^{1/4\pi\iota} \int_{\eta_0}^{\infty} e^{-1/2\pi\iota\eta^2}\, d\eta.$$

In the corresponding part of S_3 θ has the opposite sign and therefore the sum is negligible in comparison with S_{43}.

If p is the perpendicular from O on O_1P, $p = r \sin \alpha$, and when p is greater than a the point P is outside the boundary of the geometrical shadow, η_0 is negative and increases rapidly as p increases, and the above value $\overline{S}_{43}$ tends to the value found in the case where the point P was supposed not close to the boundary. When p is less than a the point P is inside the boundary of the geometrical shadow, and the above value of $-\overline{S}_{43}$ is the principal part of ψ, that is

$$\overline{\psi} = 2^{-1/2}\overline{\psi}_1 e^{1/4\pi\iota} \int_{\eta_0}^{\infty} e^{-1/2\pi\eta^2\iota}\, d\eta,$$

where η_0 has the value given above.* This result expresses the ratio of the magnetic force at a point inside the boundary of the geometrical shadow to the magnetic force due to the oscillator alone in terms of one quantity η_0. For values of η_0 greater than 5 integration by parts will give a sufficient approximation, for values of η_0 less than 5 it is convenient to write the above in the form

$$\overline{\psi} = \tfrac{1}{2}\overline{\psi}_1\,[1 - (L+M) - \iota\,(L-M)],$$

where

$$L = \int_0^{\eta_0} \cos \tfrac{1}{2}\pi\eta^2 d\eta, \qquad M = \int_0^{\eta_0} \sin \tfrac{1}{2}\pi\eta^2 d\eta,$$

and use GILBERT's tables for these integrals. An important particular case is that for which O_1 and P are both close to the surface of the sphere, as in the case of wireless telegraphy. The tables given below show how the amplitude of the oscillations diminishes as the distance along the earth's surface from the transmitter is increased. In calculating these tables the transmitter and receiver have been taken to be vertical antennæ, the fundamental wave-length is five times the height, and the results are given in two cases: for oscillations of wave-length one-fifth of a mile corresponding to antennæ of height 211 feet, and for oscillations of wave-length one-quarter of a mile corresponding to antennæ of height 264 feet. The first column gives the angular distance of the receiver from the transmitter, the second column

* The investigation assumes that the series in ζ obtained by the substitution $n = n_1 + \zeta$ converge for values of n up to z_0, and this will be so if P is near the boundary of the shadow.

the distance d of the receiver from the transmitter in miles, the third column gives the corresponding value of η_0 calculated from the formula

$$\eta_0 = 2\,(a - r\cos\tfrac{1}{2}\theta)\,(\lambda r \sin\tfrac{1}{2}\theta)^{-1/2},$$

the last column gives the ratio of the amplitude F of the oscillations at the receiver to the amplitude F_1 of the oscillations due to the transmitter alone.

θ.	d.	η_0.	F/F_1.
° ′			
1 0	70	·084	·46
1 20	93	·151	·43
1 40	116	·225	·40
2 0	140	·302	·37
2 20	163	·391	·34
2 40	186	·483	·32
3 0	209	·582	·29
3 20	233	·685	·26
3 40	256	·794	·24
4 0	279	·907	·22
4 20	302	1·025	·20
4 40	326	1·148	·18
5 0	349	1·275	·16
5 20	372	1·405	·15
5 40	396	1·542	·14
6 0	419	1·681	·13
6 20	442	1·828	·12
6 40	465	1·973	·11
7 0	489	2·124	·10
7 20	512	2·278	·09
7 40	535	2·436	·09
8 0	558	2·598	·08
8 20	582	2·763	·07
8 40	605	2·931	·07
9 0	628	3·103	·07
9 20	651	3·276	·06
λ = ·2 mile.			

θ.	d.	η_0.	F/F_1.
° ′			
1 0	70	·068	·47
1 20	93	·129	·44
1 40	116	·196	·41
2 0	140	·268	·38
2 20	163	·345	·36
2 40	186	·428	·33
3 0	209	·516	·30
3 20	233	·609	·28
3 40	256	·706	·25
4 0	279	·807	·23
4 20	302	·914	·21
4 40	326	1·024	·19
5 0	349	1·137	·18
5 20	372	1·255	·16
5 40	396	1·377	·15
6 0	419	1·501	·14
6 20	442	1·630	·13
6 40	465	1·762	·12
7 0	489	1·897	·11
7 20	512	2·035	·11
7 40	535	2·177	·10
8 0	558	2·321	·09
8 20	582	2·469	·09
8 40	605	2·619	·08
9 0	628	2·773	·08
9 20	651	2·929	·07
λ = ·25 mile.			

APPENDIX.—Investigation of Mathematical Results required.

1. *Summation of Series.*

If u_n denotes the general term of a series, and S_m the sum of m terms, the first of which is u_n,

$$S_m = u_n + u_{n+1} + \ldots + u_{n+m-1},$$

which can be written symbolically

$$S_m = (1 + e^{D} + e^{2D} + \ldots + e^{(m-1)D})\,u_{n+t},$$

where D is d/dt and t is put equal to zero after the operations are performed. This is equivalent to

$$S_m = \frac{e^{mD}-1}{e^{D}-1} u_{n+t},$$

or

$$S_m = \frac{1}{D}\frac{D}{e^{D}-1}(e^{mD}-1)\, u_{n+t},$$

which can be expressed by

$$S_m = \int_0^m \frac{D}{e^{D}-1} u_{n+t}\, dt.*$$

The present object is to replace this integral by expressions suitable for calculation in different cases. The first case is that in which u_{n+t} contains an exponential factor e^{at}, where $\alpha - 2k\pi\iota$ (k an integer) does not vanish or is not very small. Writing

$$u_{n+t} = e^{at} w_{n+t},$$

where w_{n+t} contains no other exponential factor of the form e^{at},

$$\frac{D}{e^{D}-1} u_{n+t} = \frac{D}{e^{D}-1} e^{at} w_{n+t} = e^{at}\frac{D+\alpha}{e^{D+\alpha}-1} w_{n+t},$$

that is

$$\frac{D}{e^{D}-1} u_{n+t} = e^{at} \sum_{k=0} \frac{1}{k!}\frac{d^k}{d\alpha^k}\left(\frac{\alpha}{e^{\alpha}-1}\right) D^k w_{n+t},$$

or

$$\frac{D}{e^{D}-1} u_{n+t} = e^{at} \sum_{k=0} \frac{1}{k!}\left[\alpha \frac{d^k}{d\alpha^k}\left(\frac{1}{e^{\alpha}-1}\right) + k\frac{d^{k-1}}{d\alpha^{k-1}}\left(\frac{1}{e^{\alpha}-1}\right)\right] D^k w_{n+t},$$

hence

$$\frac{D}{e^{D}-1} u_{n+t} = \sum_{k=0} \frac{1}{k!}\frac{d^k}{d\alpha^k}\left(\frac{1}{e^{\alpha}-1}\right)\left[\alpha e^{at} D^k w_{n+t} + e^{at} D^{k+1} w_{n+t}\right],$$

that is

$$\frac{D}{e^{D}-1} u_{n+t} = \sum_{k=0} \frac{1}{k!}\frac{d^k}{d\alpha^k}\left(\frac{1}{e^{\alpha}-1}\right)\frac{d}{dt}(e^{at} D^k w_{n+t}).$$

Therefore

$$S_m = \int_0^m \sum_{k=0} \frac{1}{k!}\frac{d^k}{d\alpha^k}\left(\frac{1}{e^{\alpha}-1}\right)\frac{d}{dt}(e^{at} D^k w_{n+t})\, dt,$$

whence

$$S_m = \left\{\sum_0 \frac{1}{k!}\frac{d^k}{d\alpha^k}\left(\frac{1}{1-e^{\alpha}}\right) D^k w_{n+t}\right\}_{t=0} - \left\{\sum_0 \frac{1}{k!}\frac{d^k}{d\alpha^k}\left(\frac{1}{1-e^{\alpha}}\right) e^{at} D^k w_{n+t}\right\}_{t=m}$$

The sum S_m is thus expressed as the difference of two series which, in general, are diverging series, but they can be used for approximate calculation if the last term taken is within that part of the series which converges. When the series S_m

* *Cf.* BOOLE, 'Finite Differences,' ch. v.

converges to a limit when m increases without limit, the limit S to which it converges will be given approximately* by

$$S = \left\{ \sum_0 \frac{1}{k!} \frac{d^k}{d\alpha^k} \left(\frac{1}{1-e^\alpha} \right) D^k w_{n+t} \right\}_{t=0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)},$$

it being understood that only the terms of this series where it converges are to be taken.

The above clearly fails it $\alpha = 2k\pi\iota$, for in that case $1-e^\alpha$ vanishes; it is also inapplicable if $\alpha - 2k\pi\iota$ is very small, for then $1-e^\alpha$ is very small, and the terms of (i.) diverge at once. The result shows that the sum S_m, however great m is, is not of an order higher than that of the terms that compose it when $\alpha - 2k\pi\iota$ does not vanish or is not very small; this may be compared with the known case of DIRICHLET'S integral which vanishes unless the range of integration incloses the origin.

The second case to be considered is that in which $\alpha = 0$ (to which the case $\alpha = 2k\pi\iota$ is always reducible) and u_{n+t} contains an exponential factor $e^{\beta t^2}$; writing

$$u_{n+t} = e^{\beta t^2} w_{n+t},$$

and remembering that

$$\frac{D}{e^D - 1} = 1 - \tfrac{1}{2} D + \sum_1 (-)^{k-1} \frac{1}{2k!} B_{2k-1} D^{2k},$$

where B_{2k-1} are BERNOULLI'S numbers, it follows that

$$S_m = \int_0^m e^{\beta t^2} w_{n+t} dt + \left\{ \left(-\tfrac{1}{2} + \sum_1 (-)^{k-1} \frac{1}{2k!} B_{2k-1} D^{2k-1} \right) e^{\beta t^2} w_{n+t} \right\}_{t=m}$$
$$- \left\{ \left(-\tfrac{1}{2} + \sum_1 (-)^{k-1} \frac{1}{2k!} B_{2k-1} D^{2k-1} \right) e^{\beta t^2} w_{n+t} \right\}_{t=0}.$$

When the real part of β is not greater than zero the important part of the integral $\int_0^m e^{\beta t^2} w_{n+t}\, dt$ is that contributed by small values of t, and writing

$$w_{n+t} = w_n + t w'_n + \tfrac{1}{2} t^2 w''_n + \text{\&c.},$$

it follows that

$$\int_0^m e^{\beta t^2} w_{n+t}\, dt = w_n \int_0^m e^{\beta t^2}\, dt + w'_n \int_0^m e^{\beta t^2} t\, dt + \text{\&c.}\dagger$$

Now

$$\int_0^m e^{\beta t^2}\, dt = \int_0^\infty e^{\beta t^2}\, dt - \int_m^\infty e^{\beta t^2}\, dt,$$

that is

$$\int_0^m e^{\beta t^2}\, dt = \tfrac{1}{2} \pi^{1/2} (-\beta)^{-1/2} + e^{\beta m^2}/2\beta m - \text{\&c.},$$

* When w_{n+t} only involves t as a polynomial the series has only a finite number of terms which represent the exact sum.

† For the determination of the important part of the integral it is sufficient that w_{n+t} should be expressible in powers of t for small values of t.

and therefore, m not being small, the important part of $\int_0^m e^{\beta t^2}\,dt$ is $\frac{1}{2}\pi^{1/2}(-\beta)^{-1/2}$; similarly, the important part of $\int_0^m e^{\beta t^2}t\,dt$ is $-\frac{1}{2}\beta^{-1}$; hence, the important part of $\int_0^m e^{\beta t^2}w_{n+t}\,dt$ is $\frac{1}{2}w_n\pi^{1/2}(-\beta)^{-1/2}$, provided w'_n is of lower order than $w_n\beta^{1/2}$ with corresponding conditions for w''_n, &c. Therefore, when $\beta^{-1/2}$ is of an order higher than unity, the most important part of S_m is $\frac{1}{2}w_n\pi^{1/2}(-\beta)^{-1/2}$, in this case the sum of an order higher than that of the terms that compose it.

The third case is when α is small and u_{n+t} contains an exponential factor of the form $e^{\alpha t+\beta t^2}$; as in the immediately preceding case, the result depends on the value of the integral $\int_0^m e^{\alpha t+\beta t^2}w_{n+t}\,dt$.

If the real part of β is negative, or if α and β are both pure imaginaries, the important part of this integral is $w_n\int_0^\infty e^{\alpha t+\beta t^2}\,dt$, with the same restrictions as in the previous case.

The fourth case is that when $\beta=0$, α is small, and u_{n+t} contains an exponential factor $e^{\alpha t+\gamma t^3}$; as above, the result depends on the value of the integral $\int_0^m e^{\alpha t+\gamma t^3}w_{n+t}\,dt$, and, if the real part of γ is negative, or if both α and γ are pure imaginaries, the important part of this integral is $w_n\int_0^\infty e^{\alpha t+\gamma t^3}\,dt$, with restrictions similar to those of the two preceding cases. The integrals in the third and fourth cases have been fully investigated and tabulated.

2. *Approximate Expressions for the Bessel Functions.*

Most of the expressions to be investigated in what follows have been given by L. LORENZ,* who obtained them from expressions for the sum of the squares of the two solutions and the product of the two solutions.

The investigation that follows derives them directly from the fundamental expression for a solution of BESSEL's equation and the passage from the periodic form of the expressions to the form involving real exponentials, which is insufficiently treated by LORENZ, is traced. Writing

$$z^{1/2}J_{n+1/2}(z) = 2^{1/2}\pi^{-1/2}u_n,\quad (-)^n z^{1/2}J_{-n-1/2}(z) = 2^{1/2}\pi^{-1/2}v_n,$$

and using SCHLÄFLI's† formula for J_n, it follows that

$$v_n-\iota u_n = 2^{-1/2}\pi^{-1/2}z^{1/2}\Big[\int_0^\pi[(-)^n\cos\{(n+\tfrac{1}{2})\theta+z\sin\theta\}-\iota\cos\{(n+\tfrac{1}{2})\theta-z\sin\theta\}]\,d\theta$$
$$+(-)^n\sin(n+\tfrac{1}{2})\pi\int_0^\infty e^{-z\sinh\psi+(n+1/2)\psi}\,d\psi-\iota\sin(n+\tfrac{1}{2})\pi\int_0^\infty e^{-z\sinh\psi-(n+1/2)\psi}\,d\psi\Big].‡$$

* 'Œuvres Scientifiques,' tome I., p. 405.

† Math. Ann. Band III, p. 143.

‡ In what follows n will be taken to be a positive integer, and u and v will be written for u_n and v_n.

Hence

$$v-\iota u = 2^{-1/2}\pi^{-1/2}z^{1/2}\Big[-\iota\int_0^{\pi} e^{-\iota z\sin\theta+(n+1/2)\theta\iota}\,d\theta+\int_0^{\infty} e^{-z\sinh\psi+(n+1/2)\psi}\,d\psi$$
$$-\iota\sin\left(n+\tfrac{1}{2}\right)\pi\int_0^{\infty} e^{-z\sinh\psi-(n+1/2)\psi}\,d\psi\Big].$$

It is required to find approximations for the integrals above when z is large. Taking first the case where $n+\frac{1}{2}$ is less than z, the important part of $v-\iota u$ arises from the first integral on the right-hand side, and the most important part of that integral is contributed by the values of θ for which the exponent is nearly stationary; writing

$$w = -z\sin\theta+\left(n+\tfrac{1}{2}\right)\theta,$$

w is stationary when $z\cos\theta = n+\frac{1}{2}$, and putting $n+\frac{1}{2} = z\sin\alpha$, the corresponding value of θ is $\frac{1}{2}\pi-\alpha$. Hence substituting $\theta = \frac{1}{2}\pi-\alpha+\vartheta$,

$$w = -z\cos\alpha+\left(n+\tfrac{1}{2}\right)\left(\tfrac{1}{2}\pi-\alpha\right)+\tfrac{1}{2}z\cos\alpha\vartheta^2+\frac{1}{3\,!}z\sin\alpha\vartheta^3+,\ \&c.,$$

and

$$\int_0^{\pi} e^{-\iota z\sin\theta+(n+1/2)\iota\theta}d\theta = \int_{\alpha-1/2\pi}^{\alpha+1/2\pi} e^{-\iota z\cos\alpha+(n+1/2)(1/2\pi-\alpha)\iota+1/2z\cos\alpha\vartheta^2\iota+1/6z\sin\alpha\theta^3}\,d\vartheta\,;$$

that is

$$\int_0^{\pi} e^{-\iota z\sin\theta+(n+1/2)\iota\theta}\,d\theta = e^{-\iota z\cos\alpha+(n+1/2)(1/2\pi-\alpha)\iota}\int_{\alpha-1/2\pi}^{\alpha+1/2\pi} e^{1/2z\iota\cos\alpha\vartheta^2+1/6z\iota\sin\alpha\vartheta^3+\dots}\,d\vartheta.$$

Now unless $\frac{1}{2}\pi-\alpha$ is small, the important part of the integral on the right-hand side is

$$\int_{\alpha-1/2\pi}^{\alpha+1/2\pi} e^{1/2z\iota\cos\alpha\vartheta^2}\,d\vartheta = \int_{-\infty}^{\infty} e^{1/2z\iota\cos\alpha\vartheta^2}\,d\vartheta-\int_{\alpha+1/2\pi}^{\infty} e^{1/2z\iota\cos\alpha\vartheta^2}\,d\vartheta-\int_{-\infty}^{\alpha-1/2\pi} e^{1/2z\iota\cos\alpha\vartheta^2}\,d\vartheta,$$

and unless $\{(\frac{1}{2}\pi-\alpha)\,z\cos\alpha\}^{-1}$ is of the same or of higher order than $(z\cos\alpha)^{-1/2}$, the second and third integrals on the right are negligible in comparison with the first, that is, if $z-n-\frac{1}{2}$ is of an order higher than $z^{1/3}$,

$$\int_{\alpha-1/2\pi}^{\alpha+1/2\pi} e^{1/2z\iota\cos\alpha\vartheta^2}\,d\vartheta = 2^{1/2}\pi^{1/2}(-\iota z\cos\alpha)^{-1/2} = 2^{1/2}\pi^{1/2}(z\cos\alpha)^{-1/2}\,e^{1/4\pi\iota},$$

and therefore, with the same restriction as to the magnitude of n,

$$\int_0^{\pi} e^{-\iota z\sin\theta+(n+1/2)\iota\theta}\,d\theta = 2^{1/2}\pi^{1/2}(z\cos\alpha)^{-1/2}\,e^{\iota[-z\cos\alpha+(n+1)1/2\pi-(n+1/2)\alpha]},$$

the term of highest order only being retained. Hence

$$v-\iota u = (\cos\alpha)^{-1/2}\,e^{\iota[-z\cos\alpha+1/2n\pi-(n+1/2)\alpha]},^{*}$$

* The parts contributed by the second and third integrals in the first expression for $v-\iota u$ are of order z^{-1} and therefore negligible.

when $z-n-\frac{1}{2}$ is of an order higher than $z^{1/3}$, and this is equivalent to

$$u = \mathrm{R}^{1/2}\sin\phi, \qquad v = \mathrm{R}^{1/2}\cos\phi,$$

where

$$\mathrm{R} = \sec\alpha \text{ and } \phi = z\cos\alpha - \tfrac{1}{2}n\pi + (n+\tfrac{1}{2})\,\alpha \quad . \quad . \quad . \quad . \quad . \quad (\text{ii.}).$$

When $z-n-\frac{1}{2}$ is of the same or of lower order than $z^{1/3}$, the part of the first integral in the expression for $v-\iota u$ that is most important is that contributed by small values of θ; the second integral now becomes of the same order, and the most important part of it is contributed by values of ψ near to zero; hence

$$v-\iota u = 2^{-1/2}\pi^{-1/2}z^{1/2}\left[-\iota\int_0^{\theta_0} e^{-\iota[(z-n-1/2)\,\theta-1/6z\theta^3]}\,d\theta + \int_0^{\psi_0} e^{-(z-n-1/2)\,\psi-1/6z\psi^3}\,d\psi\right],$$

the remaining parts being of lower order, and therefore writing

$$\theta = 6^{1/3}z^{-1/3}e^{1/6\pi\iota}\zeta, \quad \psi = 6^{1/3}z^{-1/3}\zeta, \quad 6^{1/3}z^{-1/3}\,(n+\tfrac{1}{2}-z) = 3\mu,$$

it follows that

$$v-\iota u = 2^{-1/2}6^{1/3}\pi^{-1/2}z^{1/6}\left[\int_0^{\zeta_0} e^{3\mu\zeta-\zeta^3}\,d\zeta + e^{-1/3\pi\iota}\int_0^{\zeta_1} e^{-3\mu e^{-1/3\pi\iota}\zeta-\zeta^3}\,d\zeta\right],$$

where ζ_0 and ζ_1 are large quantities, the first being proportional to $2^{1/3}\theta_0$ and the second to $2^{1/3}\psi_0$, and as only the most important part of $v-\iota u$ is required, and the parts contributed by large values of ζ are negligible in comparison, ζ_0 and ζ_1 may be replaced by ∞. Hence all but the parts of highest order being neglected

$$v-\iota u = 2^{-1/2}6^{1/3}\pi^{-1/2}z^{1/6}\left[\int_0^{\infty} e^{3\mu\zeta-\zeta^3}\,d\zeta + e^{-1/3\pi\iota}\int_0^{\infty} e^{-3\mu e^{-1/3\pi\iota}\zeta-\zeta^3}\,d\zeta\right] \quad . \quad . \quad . \quad (\text{iii}).*$$

The values of R and ϕ in this case have now to be obtained, and it is convenient to take first the case where $|n+\frac{1}{2}+z|$ is small compared with $z^{1/3}$. The expression in (iii) can for this purpose be expanded in ascending powers of μ which gives

$$v-\iota u = 2^{-1/2}6^{1/3}\pi^{-1/2}z^{1/6}\sum_{k=0}\frac{(3\mu)^k}{\Pi(k)}\left[1+e^{-1/3\pi\iota}(-e^{-1/3\pi\iota})^k\right]\int_0^{\infty} e^{-\zeta^3}\zeta^k\,d\zeta,$$

that is

$$v-\iota u = 2^{-1/2}6^{1/3}\pi^{-1/2}z^{1/6}\sum_{k=0}\left(1+e^{1/3(2k-1)\pi\iota}\right)\tfrac{1}{3}\,(3\mu)^k\,\frac{\Pi\{\frac{1}{3}(k-2)\}}{\Pi(k)}.$$

For the purposes required it is sufficient to know the values of R and ϕ to the second power of μ, hence neglecting μ^3 and higher powers of μ,

$$v = 2^{5/6}3^{-2/3}\pi^{-1/2}z^{1/6}\left[\Pi(-\tfrac{2}{3})+3\mu\Pi(-\tfrac{1}{3})\right]\cos^2\tfrac{1}{6}\pi,$$

$$u = 2^{5/6}3^{-2/3}\pi^{-1/2}z^{1/6}\left[\Pi(-\tfrac{2}{3})-3\mu\Pi(-\tfrac{1}{3})\right]\sin\tfrac{1}{6}\pi\cos\tfrac{1}{6}\pi;$$

* It can be verified that this integral is a solution of the differential equation $\frac{d^2y}{d\mu^2} - 9\mu y = 0$, which is approximately BESSEL'S equation for $r^{-1/3}\mathrm{K}_{n+1/2}$ when $|n+\frac{1}{2}-z|$ is of the same or lower order than $z^{1/3}$.

therefore

$$\mathrm{R} = 2^{5/3}3^{-4/3}\pi^{-1}z^{1/3}\left[\{\Pi(-\tfrac{2}{3})\}^2+6\mu\Pi(-\tfrac{1}{3})\,\Pi(-\tfrac{2}{3})\cos\tfrac{1}{3}\pi+9\mu^2\{\Pi(-\tfrac{1}{3})\}^2\right]\cos^2\tfrac{1}{6}\pi.$$

Now,

$$\Pi(-\tfrac{2}{3}) = 2^{-2/3}\pi^{-1/2}\Pi(-\tfrac{1}{3})\,\Pi(-\tfrac{5}{6}),$$

whence

$$\{\Pi(-\tfrac{2}{3})\}^2 = 2^{1/3}3^{-1/2}\pi^{1/2}\Pi(-\tfrac{5}{6}),$$

similarly,

$$\{\Pi(-\tfrac{1}{3})\}^2 = 2^{2/3}3^{-1/2}\pi^{1/2}\Pi(-\tfrac{1}{6});$$

hence

$$\mathrm{R} = 3^{-5/6}\pi^{-1/2}z^{1/3}\left[\Pi(-\tfrac{5}{6})+2^{2/3}\Pi(-\tfrac{1}{2})\,.\,3\mu+2^{1/3}\Pi(-\tfrac{1}{6})(3\mu)^2\right]. \quad . \quad . \quad \text{(iv.)}.$$

Again,

$$\tan\phi = \frac{\Pi(-\tfrac{2}{3})-3\mu\Pi(-\tfrac{1}{3})}{\Pi(-\tfrac{2}{3})+3\mu\Pi(-\tfrac{1}{3})}\tan\tfrac{1}{6}\pi,$$

therefore $\phi = \tfrac{1}{6}\pi-\beta$, where β is given by

$$\tan\beta = \frac{3\mu\sin\tfrac{1}{3}\pi\Pi(-\tfrac{1}{3})}{\Pi(-\tfrac{2}{3})+\tfrac{3}{2}\mu\Pi(-\tfrac{1}{3})},$$

whence

$$\beta = 3\mu c\,(1-\tfrac{3}{2}\mu c)\sin\tfrac{1}{3}\pi,$$

where

$$c = \Pi(-\tfrac{1}{3})/\Pi(-\tfrac{2}{3}) = 2^{-1/3}\pi^{-1/2}\Pi(-\tfrac{1}{6}),$$

and

$$\phi = \tfrac{1}{6}\pi-3\mu c\,(1-\tfrac{3}{2}\mu c)\sin\tfrac{1}{3}\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v.)}.^*$$

The value of χ, where

$$\tan\chi = -\tfrac{1}{2}\frac{d\mathrm{R}}{dz}$$

is also required; it is, however, more convenient to calculate $\phi+\chi$. From the relations

$$-\tfrac{1}{2}\frac{d\mathrm{R}}{dz} = u\frac{du}{dz}+v\frac{dv}{dz}, \qquad v\frac{du}{dz}-u\frac{dv}{dz} = 1,$$

it follows that

$$\tan\chi = -\frac{uu'+vv'}{u'v-uv'}.$$

Now

$$\tan\phi = \frac{u}{v},$$

therefore

$$\frac{v'}{u'} = \frac{\tan\phi+\tan\chi}{\tan\phi\tan\chi-1}, \qquad \text{that is } \tan(\phi+\chi) = -\frac{v'}{u'};$$

to obtain $\phi+\chi$ to the second power of μ, the third power of μ must be retained in u and v, hence

$$\tan(\phi+\chi) = \frac{\{\Pi(-\tfrac{1}{3})+\tfrac{1}{2}\Pi(\tfrac{1}{3})(3\mu)^2\}\cos^2\tfrac{1}{6}\pi}{\Pi(-\tfrac{1}{3})\sin\tfrac{1}{6}\pi\cos\tfrac{1}{6}\pi+\tfrac{1}{2}\Pi(\tfrac{1}{3})(3\mu)^2\sin\tfrac{5}{6}\pi\cos\tfrac{5}{6}\pi},$$

* These expressions for R and ϕ are to the order given, the same as those given by LORENZ.

that is

$$\tan(\phi+\chi) = \frac{\Pi\left(-\frac{1}{3}\right)+\frac{1}{2}\Pi\left(\frac{1}{3}\right)(3\mu)^2}{\Pi\left(-\frac{1}{3}\right)-\frac{1}{2}\Pi\left(\frac{1}{3}\right)(3\mu)^2}\tan\tfrac{1}{3}\pi,$$

therefore $\phi+\chi = \frac{1}{3}\pi+\gamma$ where

$$\tan\gamma = \frac{\Pi\left(\frac{1}{3}\right)(3\mu)^2\tan\frac{1}{3}\pi}{\Pi\left(-\frac{1}{3}\right)\sec^2\frac{1}{3}\pi}, \qquad \text{that is } \gamma = \frac{1}{4c}3^{-1/2}(3\mu)^2,$$

whence

$$\phi+\chi = \tfrac{1}{3}\pi+\frac{1}{4c}3^{-1/2}(3\mu)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vi.)}.$$

When $z-n-\frac{1}{2}$ is of the same order as $z^{1/3}$, the series for $v-\iota u$ in ascending powers of μ is not suitable for obtaining approximate values, and must therefore be replaced by a series involving inverse powers of μ. To effect this it is necessary to obtain the principal part of the integrals in (iii.); as in this case μ is negative, the principal part is contributed by the second integral. Writing

$$w = -3\mu e^{-1/3\pi\iota}\zeta-\zeta^3,$$

the important part of the integral arises from values of ζ in the neighbourhood of the value of ζ that makes w stationary; this value is given by $\xi = (-\mu)^{1/2}e^{-1/6\pi\iota}$, and substituting

$$\zeta = (-\mu)^{1/2}e^{-1/6\pi\iota}+\zeta,$$

it follows that

$$e^{-1/3\pi\iota}\int_0^\infty e^{-3\mu e^{-1/3\pi\iota}\zeta-\zeta^3}d\zeta = e^{-2\iota(-\mu)^{3/2}-1/3\pi\iota}\int_{-(-\mu)^{1/2}e^{-1/6\pi\iota}}^{\infty e^{-1/6\pi\iota}} e^{-3(-\mu)^{1/2}e^{-1/6\pi\iota}\zeta_1{}^2-\zeta_1{}^3}d\zeta;$$

when μ is not small the principal part of the integral on the right-hand side is the same as that of the integral whose lower limit is $-\infty\, e^{-1/6\pi\iota}$, and therefore the principal part only being retained,

$$e^{-1/3\pi\iota}\int_0^\infty e^{-3\mu e^{-1/3\pi\iota}\zeta-\zeta^3}d\zeta = 3^{-1/2}\pi^{1/2}(-\mu)^{-1/4}e^{-2\iota(-\mu)^{3/2}-1/4\pi\iota}.$$

Hence, the principal part only being retained,

$$v-\iota u = 6^{-1/6}z^{1/6}(-\mu)^{-1/4}e^{-2\iota(-\mu)^{3/2}-1/4\pi\iota};$$

therefore

$$\mathrm{R} = 6^{-1/3}z^{1/3}(-\mu)^{-1/2}, \qquad \phi = 2(-\mu)^{3/2}+\tfrac{1}{4}\pi,$$

and substituting for μ its value, this becomes

$$\mathrm{R} = z^{1/2}\left\{2\left(z-n-\tfrac{1}{2}\right)\right\}^{-1/2}, \qquad \phi = \tfrac{1}{3}2^{3/2}\left(z-n-\tfrac{1}{2}\right)^{3/2}z^{-1/2}+\tfrac{1}{4}\pi \quad . \quad . \quad . \quad \text{(vii.)}.$$

These are the forms that the expressions for R and ϕ, when $z-n-\frac{1}{2}$ is of higher order than $z^{1/3}$, take when α is near to $\frac{1}{2}\pi$; for, writing $\frac{1}{2}\pi-\alpha=\epsilon$, it follows that

$$\phi = z\cos\alpha-\tfrac{1}{2}n\pi+\left(n+\tfrac{1}{2}\right)\alpha$$

becomes

$$\phi = z \sin \epsilon - (n+\tfrac{1}{2})\,\epsilon + \tfrac{1}{4}\pi\,;$$

now

$$z-n-\tfrac{1}{2} = z(1-\cos\epsilon) = \tfrac{1}{2}z\epsilon^2,$$

therefore

$$\phi = \tfrac{1}{3}z\epsilon^3 + \tfrac{1}{4}\pi = \tfrac{1}{3}\,2^{3/2}(z-n-\tfrac{1}{2})^{3/2}z^{-1/2} + \tfrac{1}{4}\pi$$

and

$$\mathrm{R} = z\,\{z^2-(n+\tfrac{1}{2})^2\}^{-1/2} = z\,\{z-n-\tfrac{1}{2}\}^{-1/2}\{2z-(z-n-\tfrac{1}{2})\}^{-1/2}\,;$$

that is, retaining the principal part only,

$$\mathrm{R} = z^{1/2}\{2(z-n-\tfrac{1}{2})\}^{-1/2}.$$

The leading term in $(v-\iota u)\,z^{-1/6}$ being known, a further approximation can be obtained from the differential equation

$$\frac{d^2z}{d\mu^2} - 9\mu y = 0,$$

the result is

$$v-\iota u = 6^{-1/6}z^{-1/6}(-\mu)^{-1/4}e^{-2\iota(-\mu)^{3/2}-1/4\pi\iota}\left\{1+\frac{1\,.\,5}{144}\cdot\frac{\iota}{(-\mu)^{3/2}} - \frac{1\,.\,5\,.\,7\,.\,11}{2\,.\,144^2}\cdot\frac{1}{(-\mu)^3},\ \&c.\right\}.*$$

The same remark applies to the approximation for $v-\iota u$ when $z-n-\frac{1}{2}$ is of higher order than $z^{1/3}$, the ordinary differential equation for BESSEL's functions being made use of.

When $n+\frac{1}{2}-z$ is of the same order as $z^{1/3}$ the corresponding series when μ is positive is required. The principal part of $v-\iota u$ now arises from the integral $\int_0^\infty e^{3\mu\zeta-\zeta^3}\,d\zeta$, the principal part of which is contributed by values in the neighbourhood of the value of ζ that makes $3\mu\zeta-\zeta^3$ stationary; that is, in the neighbourhood of $\zeta = \mu^{1/2}$, and writing

$$\zeta = \mu^{1/2}+\zeta_1,$$

it follows that

$$\int_0^\infty e^{3\mu\zeta-\zeta^3}\,d\zeta = e^{2\mu^{3/2}}\int_{-\mu^{1/2}}^\infty e^{-3\mu^{1/2}\zeta_1^2-\zeta_1^3}\,d\zeta_1,$$

therefore the principal part is

$$e^{2\mu^{3/2}}\int_{-\infty}^\infty e^{-3\mu^{1/2}\zeta_1^2}\,d\zeta_1 = 3^{-1/2}\pi^{1/2}\mu^{-1/4}e^{2\mu^{3/2}}.$$

This result gives the leading term in the value of v, to obtain the leading term in the value of u it is necessary to calculate the principal part of the imaginary terms, that is, the principal part of

$$\tfrac{1}{2}e^{-1/3\pi\iota}\int_0^\infty e^{-3\mu e^{-1/3\pi\iota}\zeta-\zeta^3}\,d\zeta - \tfrac{1}{2}e^{1/3\pi\iota}\int_0^\infty e^{-3\mu e^{1/3\pi\iota}\zeta-\zeta^3}\,d\zeta_1.$$

* *Cf.* STOKES, 'Camb. Phil. Trans.,' vol. x., p. 105.

The principal part of the first of these integrals arises from values of ζ in the neighbourhood of $\zeta = \mu^{1/2}e^{1/3\pi\iota}$, and of the second from values of ζ in the neighbourhood of $\zeta = \mu^{1/2}e^{-1/3\pi\iota}$, hence writing in the first $\zeta = \mu^{1/2}e^{1/3\pi\iota}+\zeta_1$, and in the second $\zeta = \mu^{1/2}e^{-1/3\pi\iota}+\zeta_1$, the expression becomes

$$\tfrac{1}{2}e^{-1/3\pi\iota}\int_{-\mu^{1/2}e^{1/3\pi\iota}}^{\infty} e^{-2\mu^{3/2}-3\mu^{1/2}e^{1/3\pi\iota}\zeta_1^2-\zeta_1^3}\,d\zeta_1-\tfrac{1}{2}e^{1/3\pi\iota}\int_{-\mu^{1/2}e^{-1/3\pi\iota}}^{\infty} e^{-2\mu^{3/2}-3\mu^{1/2}e^{-1/3\pi\iota}\zeta_1^2-\zeta_1^3}\,d\zeta_1,$$

and the principal part of this expression is equal to the principal part of

$$\tfrac{1}{2}e^{-1/3\pi\iota}\int_{-\mu^{1/2}e^{1/3\pi\iota}}^{\infty} e^{-2\mu^{3/2}-3\mu^{1/2}e^{1/3\pi\iota}\zeta_1^2}\,d\zeta_1-\tfrac{1}{2}e^{1/3\pi\iota}\int_{-\mu^{1/2}e^{-1/3\pi\iota}}^{\infty} e^{-2\mu^{3/2}-3\mu^{1/2}e^{-1/3\pi\iota}\zeta_1^2}\,d\zeta_1,$$

which, writing $\zeta_1 = \eta e^{-1/6\pi\iota}$ in the first, and $\zeta_1 = \eta e^{1/6\pi\iota}$ in the second, is equal to

$$\tfrac{1}{2}e^{-1/2\pi\iota}\int_{-\mu^{1/2}\iota}^{\infty e^{1/6\pi\iota}} e^{-2\mu^{3/2}-3\mu^{1/2}\eta^2}\,d\eta-\tfrac{1}{2}e^{1/2\pi\iota}\int_{\mu^{1/2}\iota}^{\infty e^{-1/6\pi\iota}} e^{-2\mu^{3/2}-3\mu^{1/2}\eta^2}\,d\eta\,;$$

that is, equal to

$$-\tfrac{1}{2}\iota\int_{-\mu^{1/2}\iota}^{0} e^{-2\mu^{3/2}-3\mu^{1/2}\eta^2}\,d\eta-\tfrac{1}{2}\iota\int_{\mu^{1/2}\iota}^{0} e^{-2\mu^{3/2}-3\mu^{1/2}\eta^2}\,d\eta-\iota\int_{0}^{\infty} e^{-2\mu^{3/2}-3\mu^{1/2}\eta^2}\,d\eta\,;$$

now the principal parts of the first and second integrals in this expression are equal, but with opposite signs, and therefore the principal part is

$$-\iota\int_0^{\infty} e^{-2\mu^{3/2}-3\mu^{1/2}\eta^2}\,d\eta = -\tfrac{1}{2}\iota 3^{-1/2}\pi^{1/2}\mu^{-1/4}e^{-2\mu^{3/2}}.$$

Hence

$$v-\iota u = 6^{-1/6}z^{1/6}\mu^{-1/4}\left[e^{2\mu^{3/2}}-\tfrac{1}{2}\iota e^{-2\mu^{3/2}}\right];$$

and

$$\text{R} = 6^{-1/3}z^{1/3}\mu^{-1/2}\left[e^{4\mu^{3/2}}+\tfrac{1}{4}e^{-4\mu^{3/2}}\right],$$

$$\tan\phi = \tfrac{1}{2}e^{-4\mu^{3/2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{viii.}).$$

The leading terms having been determined, the approximation can be carried further by using the differential equation, and the result is

$$v-\iota u = 6^{-1/6}z^{1/6}\mu^{-1/4}\Big[e^{2\mu^{3/2}}\Big\{1+\frac{1\,.\,5}{144}\mu^{-3/2}+\frac{1\,.\,5\,.\,7\,.\,11}{1\,.\,2\,.\,144^2}\mu^{-3}+\text{\&c.}\Big\}$$
$$-\tfrac{1}{2}\iota e^{-2\mu^{3/2}}\Big\{1-\frac{1\,.\,5}{144}\mu^{-3/2}+\frac{1\,.\,5\,.\,7\,.\,11}{1\,.\,2\,.\,144^2}\mu^{-3}-\text{\&c.}\Big\}\Big].*$$

* It should be observed that the constant of the imaginary part is half the value that would have been given by STOKES' rule, p. 112 of the paper referred to above; the explanation of this is that the value of the θ in STOKES' investigation that corresponds to this solution is one that belongs to a boundary for the intervals of θ; this case is not discussed by STOKES, but it is not difficult to prove that for such values the constant takes half the value it has in the interval.

The value of χ is given by the relations

$$\tan(\phi+\chi) = 2e^{4\mu^{3/2}}, \qquad \text{or} \qquad \tan\chi = e^{4\mu^{3/2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ix.)},$$

the leading term only being retained.

When $n+\frac{1}{2}-z$ is great compared with $z^{1/3}$, the necessary approximations are obtained from the original formula for $v-\iota u$, viz. :—

$$v-\iota u = 2^{-1/2}\pi^{-1/2}z^{1/2}\left[-\iota\int_0^{\pi} e^{-\iota z\sin\theta+(n+1/2)\theta\iota}\,d\theta+\int_0^{\infty} e^{-z\sinh\psi+(n+1/2)\psi}\,d\psi \right.$$
$$\left. -\iota\sin\left(n+\tfrac{1}{2}\right)\pi\int_0^{\infty} e^{-z\sinh\psi-(n+1/2)\psi}\,d\psi\right].$$

The principal part of this expression arises from the second integral, and is contributed by the values of ψ near to the value that makes $z\sinh\psi-(n+\frac{1}{2})\psi$ stationary; if this value of ψ is δ, then

$$z\cosh\delta = n+\tfrac{1}{2},$$

and writing $\psi = \delta+\psi_1$, it follows that

$$\int_0^{\infty} e^{-z\sinh\psi+(n+1/2)\psi}\,d\psi = \int_{-\delta}^{\infty} e^{(n+1/2)\delta-z\sinh\delta-2z\sinh\delta\sinh^2 1/2\psi_1-(n+1/2)(\sinh\psi_1-\psi_1)}\,d\psi_1,$$

the principal part of which is equal to

$$e^{(n+1/2)\delta-z\sinh\delta}\int_{-\infty}^{\infty} e^{-1/2 z\sinh\delta\psi_1^2}\,d\psi_1 = 2^{1/2}\pi^{1/2}z^{-1/2}(\sinh\delta)^{-1/2}e^{(n+1/2)\delta-z\sinh\delta},$$

the part which depends on the integral $\int_{-\infty}^{-\delta}$ being of lower order when $n+\frac{1}{2}-z$ is of higher order than $z^{1/3}$; hence

$$v = (\sinh\delta)^{-1/2}e^{(n+1/2)\delta-z\sinh\delta},$$

where δ is given by $z\cosh\delta = n+\frac{1}{2}$.

To obtain u it is necessary to calculate the leading terms of the imaginary part of the expression for $v-\iota u$, and this arises from the first integral; the part required is the principal part of

$$-\tfrac{1}{2}\iota\int_0^{\pi} e^{-\iota z\sin\theta+(n+1/2)\iota\theta}\,d\theta-\tfrac{1}{2}\iota\int_0^{\pi} e^{\iota z\sin\theta-(n+1/2)\iota\theta}\,d\theta.$$

The exponent in the first of these is stationary when $\theta = \iota\delta$, and the exponent in the second when $\theta = -\iota\delta$, writing in the first $\theta = \iota\delta-\theta_1$ and in the second $\theta = \iota\delta+\theta_1$, the expression becomes

$$-\tfrac{1}{2}\iota e^{-(n+1/2)\delta+z\sinh\delta}\left\{\int_{-\iota\delta}^{\pi-\iota\delta} e^{-2z\sinh\delta\sin^2 1/2\theta_1+\iota z\cosh\delta(\theta_1-\sin\theta_1)}\,d\theta_1+\int_{\iota\delta}^{\pi+\iota\delta} e^{-2z\sinh\delta\sin^2 1/2\theta_1-\iota z\cosh\delta(\theta_1-\sin\theta_1)}\,d\theta_1\right\},$$

the principal part of which is equal to the principal part of

$$-\tfrac{1}{2}\iota e^{-(n+\frac{1}{2})\delta+z\sinh\delta}\left\{\int_{-\iota\delta}^{\pi-\iota\delta} e^{-2z\sinh\delta\sin^2\frac{1}{2}\theta_1}\,d\theta_1+\int_{\iota\delta}^{\pi+\iota\delta} e^{-2z\sinh\delta\sin^2\frac{1}{2}\theta_1}\,d\theta_1\right\},$$

that is to the principal part of

$$-\tfrac{1}{2}\iota e^{-(n+\frac{1}{2})\delta+z\sinh\delta}\left\{\int_{-\iota\delta}^{0} e^{-2z\sinh\delta\sin^2\frac{1}{2}\theta_1}\,d\theta_1+\int_{\iota\delta}^{0} e^{-2z\sinh\delta\sin^2\frac{1}{2}\theta_1}\,d\theta_1\right.$$
$$\left.+2\int_0^{\pi} e^{-2z\sinh\delta\sin^2\frac{1}{2}\theta_1}\,d\theta_1+\int_{\pi}^{\pi-\iota\delta} e^{-2z\sinh\delta\sin^2\frac{1}{2}\theta_1}\,d\theta_1+\int_{\pi}^{\pi+\iota\delta} e^{-2z\sinh\delta\sin^2\frac{1}{2}\theta_1}\,d\theta_1\right\},$$

and this is equal to the principal part of

$$-\iota e^{-(n+\frac{1}{2})\delta+z\sinh\delta}\int_0^{\pi} e^{-2z\sinh\delta\sin^2\frac{1}{2}\theta_1}\,d\theta_1,$$

which is

$$-\iota e^{-(n+\frac{1}{2})\delta+z\sinh\delta}\int_0^{\infty} e^{-\frac{1}{2}z\theta_1^2\sinh\delta}\,d\theta_1=-2^{-\frac{1}{2}}\pi^{\frac{1}{2}}z^{-\frac{1}{2}}(\sinh\delta)^{-\frac{1}{2}}\iota e^{-(n+\frac{1}{2})\delta+z\sinh\delta},$$

and therefore

$$u=\tfrac{1}{2}(\sinh\delta)^{-\frac{1}{2}}e^{-(n+\frac{1}{2})\delta+z\sinh\delta}.$$

Hence, writing $\tau=z\sinh\delta-(n+\frac{1}{2})\delta$ where $z\cosh\delta=n+\frac{1}{2}$, we have

$$u=\tfrac{1}{2}(\sinh\delta)^{-\frac{1}{2}}e^{\tau},\qquad v=(\sinh\delta)^{-\frac{1}{2}}e^{-\tau},$$

$$R=(\sinh\delta)^{-1}\{e^{-2\tau}+\tfrac{1}{4}e^{2\tau}\},$$

$$\tan\phi=\tfrac{1}{2}e^{2\tau}\quad\ldots\ldots\ldots\ldots\quad\text{(x.)}.$$

It remains to prove that as δ becomes small these expressions become identical with those obtained for the case when $n+\frac{1}{2}-z$ is of the same order as $z^{\frac{1}{3}}$. When δ is small

$$z\sinh\delta-(n+\tfrac{1}{2})\delta=z\sinh\delta-\delta z\cosh\delta,$$

that is,

$$\tau=-\tfrac{1}{3}z\,\delta^3=-\tfrac{1}{3}\left\{\frac{2(n+\frac{1}{2}-z)}{z^{\frac{1}{3}}}\right\}^{\frac{3}{2}}.$$

Now

$$2\mu^{\frac{3}{2}}=2\left\{\left(\tfrac{6}{2}\right)^{\frac{1}{3}}\frac{(n+\frac{1}{2}-z)}{3}\right\}^{\frac{3}{2}}=\tfrac{1}{3}\left\{\frac{2(n+\frac{1}{2}-z)}{z^{\frac{1}{3}}}\right\}^{\frac{3}{2}},$$

and

$$6^{-\frac{1}{6}}z^{\frac{1}{6}}\mu^{-\frac{1}{4}}=2^{-\frac{1}{4}}z^{\frac{1}{4}}(n+\tfrac{1}{2}-z)^{-\frac{1}{4}},$$

which is $\delta^{-\frac{1}{2}}$ when δ is small, and therefore as δ diminishes, the form of the expression in (x.) becomes that in (viii.). The approximations in (x.) can, as before, be carried further by using the differential equation satisfied by $z^{-1/2}(v-\iota u)$, which is BESSEL'S equation.*

The various approximations are collected together below for convenience of reference :—

When $z-n-\frac{1}{2}$ is great compared with $z^{1/3}$,

$$R = \sec\alpha, \quad \phi = z\cos\alpha - \tfrac{1}{2}n\pi + (n+\tfrac{1}{2})\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{ii.}),$$

where $z\sin\alpha = n+\frac{1}{2}$.

When $z-n-\frac{1}{2}$ is of the same order as $z^{1/3}$,

$$R = z^{1/2}\{2(z-n-\tfrac{1}{2})\}^{-1/2}, \quad \phi = \tfrac{1}{3}2^{3/2}z^{-1/2}(z-n-\tfrac{1}{2})^{3/2}+\tfrac{1}{4}\pi \quad . \quad . \quad . \quad (\text{vii.}).$$

When $|z-n-\frac{1}{2}|$ is of lower order than $z^{1/3}$,

$$R = 3^{-5/6}\pi^{-1/2}z^{1/3}[\Pi(-\tfrac{5}{6})+2^{2/3}\Pi(-\tfrac{1}{2}).3\mu+2^{1/3}\Pi(-\tfrac{1}{6})(3\mu)^2\mu] \quad . \quad . \quad . \quad (\text{iv.}),$$

$$\phi = \tfrac{1}{6}\pi - 3\mu c(1-\tfrac{3}{2}\mu c)\sin\frac{\pi}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{v.}),$$

$$\phi+\chi = \tfrac{1}{3}\pi + \frac{1}{4c}3^{-1/2}(3\mu)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{vi.}),$$

where

$$3\mu = 6^{1/3}z^{-1/3}(n+\tfrac{1}{2}-z), \quad c = 2^{-1/3}\pi^{-1/2}\Pi(-\tfrac{1}{6}).$$

When $n+\frac{1}{2}-z$ is of the same order as $z^{1/3}$,

$$R = 6^{-1/3}z^{1/3}\mu^{-1/4}[e^{4\mu^{3/2}}+\tfrac{1}{4}e^{-4\mu^{3/2}}], \quad \tan\phi = \tfrac{1}{2}e^{-4\mu^{3/2}} \quad . \quad . \quad . \quad . \quad . \quad (\text{viii.}),$$

$$\tan\chi = e^{4\mu^{3/2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{ix.}).$$

When $n+\frac{1}{2}-z$ is great compared with $z^{1/3}$

$$R = (\sinh\delta)^{-1}[e^{-2\tau}+\tfrac{1}{4}e^{2\tau}], \quad \tan\phi = \tfrac{1}{2}e^{2\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{x.}),$$

where

$$\tau = z\sinh\delta - (n+\tfrac{1}{2})\delta, \quad z\cosh\delta = n+\tfrac{1}{2}.$$

When n is not an integer the corresponding results can be obtained by writing

* Another method of approximating to the value of R is to make use of the relation

$$R = \frac{4z}{\pi}\int_0^\infty K_0(2n\sinh\zeta)\cosh(2n+1)\zeta\,d\zeta,$$

which is not difficult to establish, and then deduce ϕ from the result; the method given is more direct, and avoids the difficulties that arise in determining the constants for the different forms of ϕ in this other method.

$e^{-n\pi\iota}$ for $(-)^n$ in the original expression, but, with a view to the special case where $n+\frac{1}{2}$ is an integer, it is preferable to use the solutions $K_{n+1/2}(\iota z)$ and $K_{n+1/2}(-\iota z)$; the expressions to be approximated to are $2^{1/2}\pi^{-1/2}z^{1/2}K_{n+1/2}(\iota z)$ and $2^{1/2}\pi^{-1/2}z^{1/2}K_{n+1/2}(-\iota z)$. With the same notation as when n is an integer

$$2^{1/2}\pi^{-1/2}z^{1/2}K_{n+1/2}(\iota z) = R^{1/2}e^{-\iota\phi-1/2(n+1/2)\pi\iota},$$

$$2^{1/2}\pi^{-1/2}z^{1/2}K_{n+1/2}(-\iota z) = R^{1/2}e^{\iota\phi+1/2(n+1/2)\pi\iota},$$

where R and ϕ have the same values as when n is an integer.

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